A randomized incremental algorithm for Delaunay triangulation in 2D

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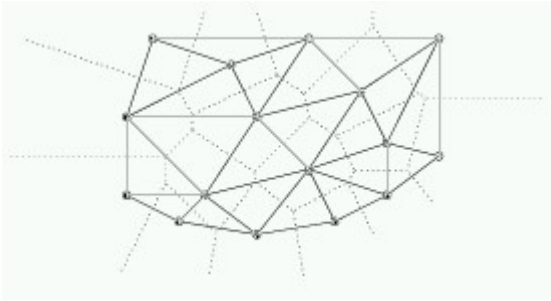
Overview:

- (shortly) introduce Delaunay triangulation
- Present EXACT incremental algorithm
- Analyze complexity + correctness

Delaunay triangulation

- most realistic triangulation for terrain perspectives
- small angles lead to unrealistic perspective: maximize minimal angle
- correspondence between big angles and empty circle property

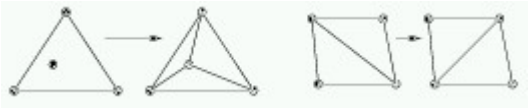
Definitions - Assumptions



- set of Sites S
- Voronoi Diagram of degree 3 $\Leftrightarrow$ no four points of S are co-circular $\Leftrightarrow$ Voronoi vertices are boundary points of exactly 3 Voronoi regions
- use fact that Voronoi vertex is center of circle C defined by 3 sites and no other sites lie in C

Methods

CCW Orientation Test and INCIRCLE Test



- CCW Orientation Test : used to decide whether a given point lies in a triangle

- INCIRCLE Test : decides whether a point lies in a circle (flip of inner edge)

CCW Orientation Test (1)

Let

- $p_1, p_2, p_3 \in S$
  $p_i = (x_i, y_i),\ i = 1, 2, 3$

- $\overline{p_1 p_2}$ line segment from p_1 to p_2

- $D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

CCW Orientation Test(2)

Then the orientation of p_3 to $\overline{p_1 p_2}$ is

- to the left of $\overline{p_1 p_2}$ if $D > 0$
- to the right of $\overline{p_1 p_2}$ if $D < 0$
- on the line through p_1, p_2 if $D = 0$

INCIRCLE Test(1)

Let

- $p_1, p_2, p_3, p_4 \in S$
  $p_i = (x_i, y_i),\ i = 1, 2, 3$

- circle C defined by p_1, p_2, p_3

- $$D = \begin{vmatrix} x_1 & y_1 & x_1^2 + y_1^2 & 1 \\ x_2 & y_2 & x_2^2 + y_2^2 & 1 \\ x_3 & y_3 & x_3^2 + y_3^2 & 1 \\ x_4 & y_4 & x_4^2 + y_4^2 & 1 \end{vmatrix}$$

INCIRCLE Test(2)

Then the orientation of p_4 to C is

- inside C if $D > 0$
- outside C if $D < 0$
- on C if $D = 0$

ESSA can do this exact!

Incremental property of Algorithm (1)

Let

- the sites $S = \{p1, \ldots, p_n\}$
- find triangle ABC with vertices $a, b, c \notin S$ that contains all points of S

Define

- $S_i = \{a, b, c, p_1, \ldots, p_i\}$
- τ_i the Delaunay triangulation of S_i $(i = 1, \ldots, n)$

Incremental property of Algorithm (2)

Construction of τ_i by

- τ_{i-1}
- using CCW Orientation Test and INCIRCLE Test

Incremental property of Algorithm (3)

Steps:

- Start with τ_0 = Delaunay triangulation of $S_0 = \{a, b, c\}$ $= ABC$
- add p_i to S_{i-1} and fix until τ_i
- End with τ_n and remove a, b, c and all edges with these endpoints

The Algorithm (1)

Input: the sites $S = \{p_1, \ldots, p_n\}$ with no 4 points cocircular
Output: the Delaunay triangulation of S

1. find 3 points $a, b, c \notin S$ with S in the triangle ABC defined by a, b, c

2. Set $i = 0$ and init $\tau_i = ABC$

3. Add p_i to τ_i and find a triangle of τ_i, say u, v, w containing p_i and no other points of S_i

The Algorithm (2)

4. if p_i **in** the triangle u, v, w
   $\tau_{i+1} = \tau_i$ + edges up_i,vp_i,wp_i

Fix surrounding edges.
$\Rightarrow$ LEGALIZEEDGE($\overline{uv}$,τ_{i+1})
$\Rightarrow$ LEGALIZEEDGE($\overline{vw}$,τ_{i+1})
$\Rightarrow$ LEGALIZEEDGE($\overline{wu}$,τ_{i+1})

The Algorithm (3)

5. if p_i **on** the triangle u, v, w say on $\overline{uv}$ and let z be the adjacent vertex on the right of $\overline{uv}$
   $\tau_{i+1} = \tau_i$ + edges wp_i,zp_i

   Fix surrounding edges.
   $\Rightarrow$ LEGALIZEEDGE($\overline{uz}$,τ_{i+1})
   $\Rightarrow$ LEGALIZEEDGE($\overline{zv}$,τ_{i+1})
   $\Rightarrow$ LEGALIZEEDGE($\overline{vw}$,τ_{i+1})
   $\Rightarrow$ LEGALIZEEDGE($\overline{wu}$,τ_{i+1})

The Algorithm (4)

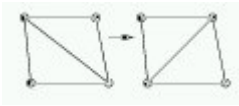
6. $i = i + 1$

7. if $i < n$ goto 3

8. discard a, b, c and all adjacent edges from τ_n

The resulting diagram is the Delaunay triangulation

How to fix surrounding edges?

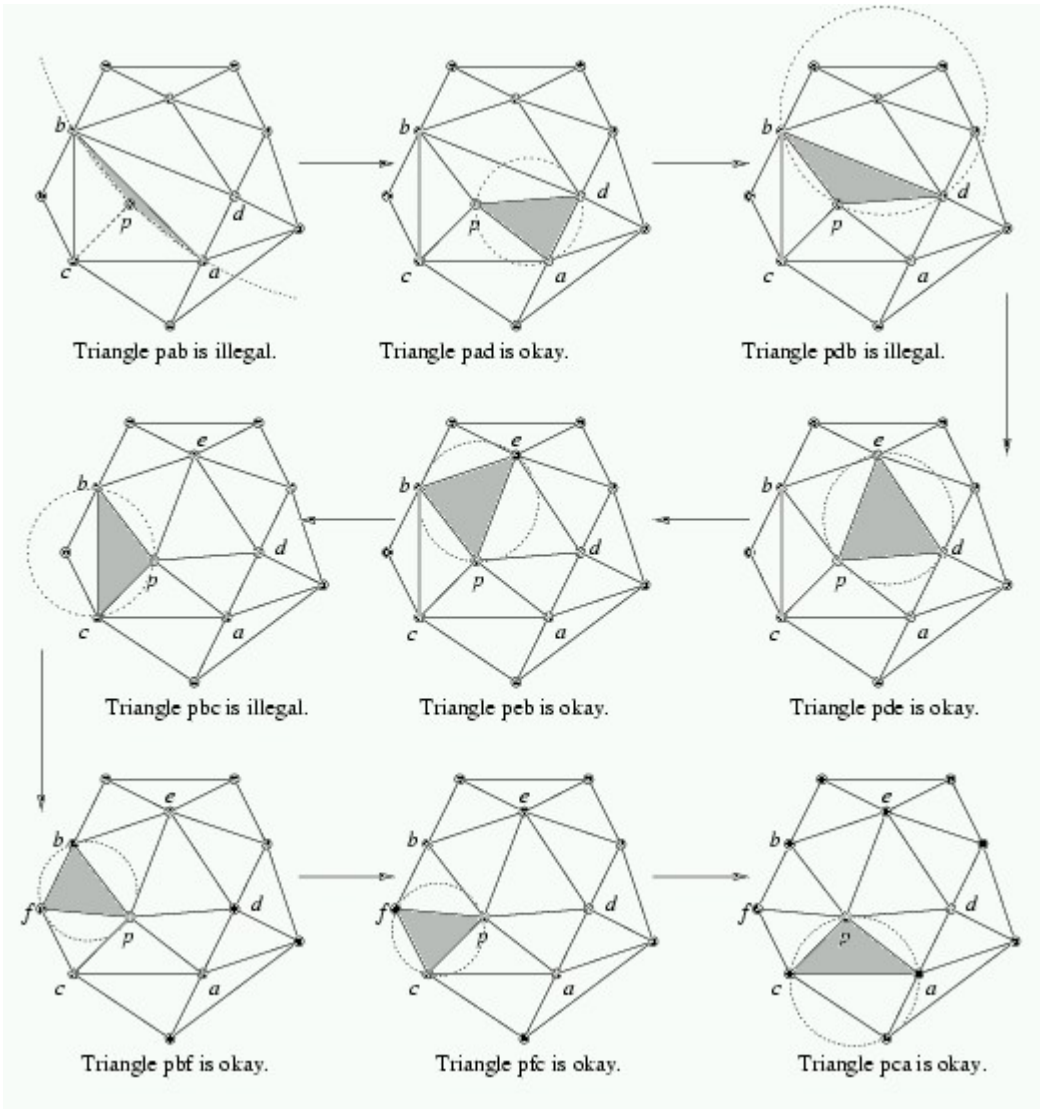


LEGALIZEEDGE$(\overline{uv}, \tau)$

1. $D =$ INCIRCLE(u, v, p_i, z)

2. if $D < 0$ (z violates the incircle test)

   - replace $\overline{uv}$ with $\overline{p_i z}$ (flipping)
   - LEGALIZEEDGE$(\overline{uz}, \tau)$
   - LEGALIZEEDGE$(\overline{zv}, \tau)$


b
p
d
c
a
Triangle pab is illegal.
Triangle pad is okay.
Triangle pdb is illegal.
e
Triangle pbc is illegal.
Triangle peb is okay.
Triangle pde is okay.
f
Triangle pbf is okay.
Triangle pfc is okay.
Triangle pca is okay.

Correctness (1)

Is LEGALIZEEDGE enough?

1. Only check edges of triangles containing the site p_i?
   $\Rightarrow p_i$ is only site that can cause a violation of the empty circle property.

2. edges inserted by algorithm are edges of Delaunay triangulation

3. Only check sites that lay on the opposite side of such an edge?
   $\Rightarrow$ violates τ_{i-1} = Delaunay triangulation

Complexity analisys (1)

- point location : O(nlogn)

- constructed triangles : $9n + 1 =$ O(n)
  - insertion of point : 3 or 4 new triangles
  - flipping : 2 new triangles
  - every edge created is an edge of Delaunay triangulation
  - Euler: max 3(r+3) -6 edges in Delaunay triangulation