A Joint Model for the Term Structure of Interest Rates and the Macroeconomy

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Abstract

We present and estimate a continuous time term structure model that incorporates both observable macroeconomic variables (output gap and inflation) and latent variables. In contrast to extant models, each of our latent factors has a macroeconomic interpretation, representing the real interest rate and the central tendencies of inflation and the real interest rate. Application to the U.S. economy shows that the model is able to describe in an accurate way the joint dynamics for the macroeconomy and the yield curve. Observable macroeconomic variables do not explain the long end of the term structure. Central tendencies of these macroeconomic variables do a much better job in this respect. These unobservable factors also play an important role in the description of the interest rate policy rule. We also find that both observable and non-observable factors determine the risk premia and hence the excess holding returns of the bonds.

Keywords: Essentially a c ne term structure model, macroeconomic factors, interest rate policy rule.
1 Introduction

Standard multi-factor term structure models identify the determinants of the yield curve in terms of a small number of state variables or factors. According to their effect on the yield curve, these are typically labeled "level", "slope" and "curvature" factors. This fully latent set-up avoids the difficulties in estimating a general equilibrium model linking the term structure to exogenous economic factors (see, for instance, Bakshi and Chen, 1996, Berardi, 2001, Buraschi and Jiltsov, 2001, and Wu, 2001). This approach, furthermore, excludes arbitrage opportunities and is econometrically and numerically tractable. It, however, does not bring us any further in understanding the macroeconomic driving forces behind the yield curve since the factors derived from such models do not have a direct economic meaning.

Recently, a tractable intermediate route for identifying the driving forces behind the term structure in terms of macroeconomic variables emerged (see Ang and Piazzesi, 2003, Piazzesi, 2003, and Fleming and Remolona, 1999). In this approach, only the necessary no-arbitrage conditions are imposed on a discrete time vector autoregressive (VAR) system containing both observable and latent factors. This framework retains from the general equilibrium approach the cross-sectional restrictions on the term structure co-movements, while, in line with the finance literature, allowing for additional latent factors. Examples of recent research in this vein include Hördahl, Tristani and Vestin (2003) and Rudebusch and Wu (2003). These papers extend the approach pioneered by Ang and Piazzesi by explicitly building on rational expectations models to construct discrete time dynamics for the factors. The mentioned papers, however, make use of latent factors without a clear macroeconomic interpretation. Dewachter and Lyrio (2003) propose a continuous time model in which this problem is overcome. These authors focus on providing a macroeconomic interpretation to the above mentioned standard finance factors. Their model builds on the results by Kozicki and Tinsley (2001, 2002), which point to the importance of long-run expectations in modeling the long end of the yield curve.

The model presented in this paper is closely related to the one presented in Dewachter and Lyrio (2003).\footnote{Dewachter et al. (2004) use a similar model to propose a benchmark against which the effects of ECB monetary policy on the German bond market can be evaluated.} We adopt an active term structure model allowing for time-varying risk premia. Our main contributions are the following. First, we allow for more flexibility in the macroeconomic dynamics. We include the central tendency of the real interest rate as one extra latent factor. Our model is then composed of two observable factors (output gap and inflation) and three latent factors (the real interest rate and the central tendencies of inflation and the real interest rate). The macroeconomic interpretation given to these latent factors distinguishes them from the standard latent factors used in the finance literature. We, therefore, refer to these factors throughout the paper as unobservable macroeconomic factors. We also allow greater flexibility in the dynamics of the central tendencies, not restricting them to random walks but adopting more general mean reversion processes. In addition, we analyze the forecast performance of the model and the importance of each macroeconomic shock on variability of the yield curve. Finally, the estimation algorithm is presented in greater detail.
That includes both the correct procedure to guarantee that both the output gap and inflation are treated as observable factors and the computation of the conditional means and variances of the factors.

The methodology adopted in this paper also presents a number of advantages when compared with the papers mentioned above (i.e. Ang and Piazzesi, 2003, Hördahl, Tristani and Vestin, 2003, and Rudebusch and Wu, 2003). First, as standard in the nance literature, we opt for a continuous time model, such that the pricing restrictions and implications are defined on any positive frequency. Second, we also do not restrict the dynamics of the system in any way. We allow the possibility of imaginary eigenvalues with respect to the spectral decomposition of the macro dynamics. For this, care is taken in the computation of the conditional means and variances of the factors. We also allow the system to be nonstationary by not restricting the dynamics of any of the factors included in the model. Third, the model is estimated in a one-step procedure. Given the macroeconomic definition of the latent factors, orthogonality between latent and observable factors no longer holds. While in the VAR literature the orthogonality among the factors was used to allow for a two-step estimation procedure, the absence of orthogonality implies that the estimation of the observable and latent factor dynamics needs to be performed in a one-step procedure. Fourth, we estimate the interest rate policy rule based on both observable and unobservable factors and, therefore, making use of information contained in the entire term structure, and not only in the short-term interest rate. Given the importance of long-term interest rates for the state of the economy, and hence for central bank behavior, a focus on the short end of the term structure seems restrictive and might be misleading. Finally, we avoid ad hoc yield curve inversion procedures.

The model is applied to U.S. data. The inclusion of two central tendencies turns out to be extremely important in fitting the yield curve and in describing the dynamics of the monetary policy rule. The time-varying risk premia are also significantly affected by these filtered factors. Furthermore, the variability of long-term bond yields is mainly explained by shocks to the inflation central tendency.

The remainder of the paper is organized as follows. In Section 2, we discuss the macroeconomic model. This model consists of a VAR dynamics in terms of both observable factors (output gap and inflation) and latent factors (real interest rate and central tendencies of inflation and the real interest rate). We also derive the bond price determination based on the proposed macro dynamics. Section 3 deals with the empirical implementation of the model. We specify in detail the Kalman filter algorithm used and the necessary expressions for the computation of the conditional means and variances of the factors. The estimation results are presented in Section 4: We first analyze the implied macro dynamics and the monetary policy implemented by the central bank. We then analyze the performance of the model in fitting the yield curve and the estimated risk premium for each bond yield. We also perform a variance decomposition analysis in order to measure the importance of macroeconomic shocks.

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2 Over the past ten years, standard interest rate policy rules have been studied extensively with mixed results. For instance, in a multi-country study Clarida, Gali and Gertler (1998) found mixed evidence for the existence of a Taylor rule.
in the variability of the yield curve. We end the section with an evaluation of the forecasting performance of the model. We conclude the paper in Section 5.

2 The model

In this section, we set out a continuous time model for the macroeconomy and the term structure of interest rates. We first outline the macroeconomic framework adopted in this paper, which is based on both observable and unobservable factors. We then analyze the implications of the macro dynamics for the term structure of interest rates. The model is part of the class of affine term structure models (ATSM)\(^3\) and is well-suited for the analysis concerning the implications of macroeconomic aggregates for the term structure of interest rates.

2.1 Dynamics of observable and unobservable macroeconomic factors

We start by presenting a stylized continuous time model for the dynamics of observable macroeconomic aggregates, i.e. the output gap \(y(t)\) and inflation \(\pi(t)\). In order to ease the empirical implementation of the model, we assume from the start that backward-looking models are good approximations to reality. The macroeconomic model is then built by the assumptions imposed on its dynamics:

\[
\begin{align*}
\text{dy}(t) &= \left[ \cdot \gamma_y y(t) + \cdot \gamma_{\pi} \left( \frac{1}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_y(t); \\
\text{d}^\frac{1}{4}(t) &= \left[ \cdot \gamma_{\pi} y(t) + \cdot \gamma_{\pi} \left( \frac{1}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi}(t); \\
\text{d}^\frac{3}{4}(t) &= \left[ \cdot \gamma_{\pi} \left( \frac{3}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi'}(t); \\
\text{d}^\frac{3}{4}(t) &= \left[ \cdot \gamma_{\pi} \left( \frac{3}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi'}(t); \\
\text{d}^\frac{3}{4}(t) &= \left[ \cdot \gamma_{\pi} \left( \frac{3}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi'}(t); \\
\text{d}^\frac{3}{4}(t) &= \left[ \cdot \gamma_{\pi} \left( \frac{3}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi'}(t); \\
\text{d}^\frac{3}{4}(t) &= \left[ \cdot \gamma_{\pi} \left( \frac{3}{4} \pi(t) \right) i \right] dt + \frac{3}{2} \gamma_i dW_{\pi'}(t);
\end{align*}
\]

where \(\frac{3}{4}\) denotes the real interest rate. The exogenous and independent central tendencies of inflation and the real interest rate are represented by \(\frac{1}{4}\) and \(\frac{3}{4}\), respectively. \(W_i(t)\); \(i = \{y; \pi; \pi'; \gamma_i\}\) denote independent Wiener processes defined on the probability space \(\langle \cdot ; F ; P \rangle\) with filtration \(F_t\). As such, we can interpret \(dW(t)\) as shocks to each of the macro variables. The dynamics for each macro variable are modeled in terms of deviations from its respective central tendency (the central tendency of the output gap is assumed equal to zero). In other words, only deviations from the central tendencies determine the short-run dynamics of the system. In this way, we actually ensure that the exogenous central tendency variables act as stochastic attractors in the system.\(^4\) The dynamics is in line with the standard

\(^3\)See Dai and Singleton (2000, 2002) and Duc and Kan (1996) for a characterization of this type of models.

\(^4\)In the estimation of the system, we impose stability of the factors and thus the attracting property of the exogenous central tendencies.
macroeconomic view. We allow each of the observable economic variables, output gap and inflation, to be affected through three channels: the instantaneous real interest rate \( \frac{1}{2} \), the other economic variables (output gap or inflation) and, finally, a mean reverting component modeling the possible inertia in the adjustment process. We also implicitly assume that the monetary authority uses a feedback rule for the real interest rate. More specifically, changes in the real interest rate \( \frac{1}{2}(t) \) are a response to deviations of the output gap or inflation from their central tendencies and to a mean reverting (real interest rate smoothing) component relative to a stochastic central tendency, \( \frac{1}{2}(t) \): The model is closed with the following definition for the nominal instantaneous interest rate \( i(t) \):

\[
i(t) = \frac{1}{2}(t) + \frac{1}{2}(t): \tag{2}
\]

The adoption of a Gaussian (Vasicek, 1977) type of model reflects our intention to offer maximal flexibility with respect to the magnitudes and sizes of the conditional and unconditional correlations among the factors. This specification fulfills the admissibility conditions specified in Dai and Singleton (2000). The costs associated with this choice are twofold: the lack of flexibility in fitting the interest rate volatility since we assume constant conditional variances for the factors; and the possibility of negative interest rates.

The above representation of the dynamics of the economy can be restated in matrix notation. Denoting \( n \) as the number of factors in the model (i.e. in our case), we define the vectors of \( n \) factors and shocks and the \( n \times n \) diagonal matrix \( S \) as:

\[
\begin{align*}
\mathbf{f}(t) & = \begin{bmatrix} y(t) \\ \frac{1}{2}(t) \\ \cdots \\ \frac{1}{2}(t) \end{bmatrix} \\
\mathbf{dW}(t) & = \begin{bmatrix} \mathbf{dW}_y(t) \\ \mathbf{dW}_{\frac{1}{2}}(t) \\ \cdots \\ \mathbf{dW}_{\frac{1}{2}}(t) \end{bmatrix} \\
\mathbf{S} & = \text{diag}(\frac{1}{2}; \frac{1}{2}; \cdots ; \frac{1}{2})
\end{align*}
\]

The dynamics of the economy can, therefore, be restated as follows:

\[
d\mathbf{f}(t) = \mathbf{K}(\mathbf{f}(t) \mid \psi) dt + \mathbf{SdW}(t); \tag{4}
\]

where

\[
\mathbf{K} = \begin{bmatrix} 2 & \psi \psi & \psi \psi & \psi \psi & \psi \psi \\ \psi \psi & \psi \psi & \psi \psi & \psi \psi & \psi \psi \\ \psi \psi & \psi \psi & \psi \psi & \psi \psi & \psi \psi \\ \psi \psi & \psi \psi & \psi \psi & \psi \psi & \psi \psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

and

\[
\psi = \mathbf{K}^{-1}(0; 0; 0; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2} ; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}).
\]

Note that since the matrix \( \mathbf{K} \) is in general not diagonal, closed form equations for the conditional mean and variance of the factors are not easily obtained. These conditional moments are needed for forecasting the evolution of the state of the economy and are discussed below in the empirical implementation of the model.
2.2 Implications for bond markets

Equation (4) completely specifies the dynamics of the macroeconomic variables and the instantaneous interest rate. This system must, therefore, also determine (up to some risk premium component) the term structure of interest rates and its dynamics. Absence of arbitrage opportunities implies that the price at time $t$ of a zero-coupon default-free bond maturing at time $T$ is defined as:

$$p(t;T) = E_t^Q \exp \int_0^t i(u) dW_A;$$

(5)

where $E_t^Q$ denotes the expectation operator under the risk-neutral probability measure $Q$. In general, this risk-neutral probability is unobserved and can only be specified by assuming some specification for the prices of factor risk. Following Duffee (2002), time variability in the prices of risk can be captured by specifying prices of risk as an affine function of the factors. The vector containing the time-varying prices of risk, $\xi$, is defined as:

$$\xi(t) = S \alpha + S^{-1} \psi f(t);$$

where $\alpha'$ $\sim (\psi; \pi; \rho; \pi', \rho')'$ and $\psi$ a $n \times n$ matrix containing the sensitivities of the prices of risk to the levels of the factors $f(t)$. Changing probability measures is then performed by means of the Girsanov theorem:

$$dW(t) = dW(t) + \xi(t) dt;$$

(6)

where $W(t)$ constitutes a martingale under measure $Q$. The dynamics under this risk-neutral metric $Q$ are given by:

$$df(t) = K \phi f(t) dt + S dW(t);$$

$$K = K + \psi;$$

(7)

$$\phi = (K + \psi)^{-1} \int_0^t K \phi f(t) S^2 \alpha \xi;$$

A functional form for bond prices can be obtained by assuming that bond prices are time homogeneous functions of the factors $f(t)$ and the time to maturity $\xi' T_i t$:

$$p(t;\xi) = p(f(t);\xi) = \exp \int_0^t a(\xi) f(t) \xi;$$

(8)

where $a(\xi)$ is a scalar and $b(\xi)$ is a $n \times 1$ vector. Imposing the no-arbitrage condition in the bond markets implies:

$$D^Q(p(f(t);\xi)) = i(t) p(f(t);\xi);$$

(9)

where $D^Q$ denotes the Dynkin operator under the probability measure $Q$. The intuitive meaning of the latter condition is that, once transformed to a risk-neutral world, instantaneous holding returns for all bonds are equal to the instantaneous riskless interest rate. Using equation (6), we can infer the implications for the real world by changing from the risk-neutral
Equations (8) and (9) determine the solution for the functions \( a(\omega) \) and \( b(\omega) \) in terms of a system of coupled ODEs that, in the general case, can only be solved numerically:

\[
\frac{\partial a(\omega)}{\partial \omega} = a_0 + \K^i \varphi_i b(\omega) \cdot \sum_{i=1}^{3} b_i^2(\omega) S_{ii}^2;
\]

\[
\frac{\partial b(\omega)}{\partial \omega} = b_0 + \K^i b(\omega);
\]

(10)

A particular solution to this system of ODEs is obtained by specifying a set of initial conditions on \( a \) and \( b \): Inspection of equation (8) shows that the relevant initial conditions are \( a(0) = 0 \) and \( b(0) = 0 \). The vectors of constants \( a_0 \) and \( b_0 \) in (10) are defined by the interest rate definition in (2) and, therefore, equal to \( a_0 = 0 \) and \( b_0 = (0 1 1 0 0)' \).

The bond pricing solution differs in important ways from the independent-factor term structure literature (see, for instance, deJong, 2000). First, allowing for interrelations among the factors (i.e., non-zero off-diagonal elements in \( \K^i \)) generates a coupled system of ODEs instead of a set of uncoupled ODEs. The bond pricing solution for the \( a \) and \( b \) functions, therefore, do not reduce to the standard multi-factor result. Second, the factor loadings no longer start from unity at maturity \( \omega = 0 \): All the factors except inflation and the real interest rate have zero loadings in the determination of the short rate.

3 Empirical implementation

We present an efficient method to estimate both the state-space dynamics, including observable and non-observable factors, as well as the prices of risk, implied by the term structure. The presence of unobservable factors requires the use of some incorporating procedure to recover the time series of these factors. In order to avoid ad hoc yield curve inversion procedures (Pearson and Sun, 1994, Chen and Scott, 1993), we opt for a Kalman filter algorithm.\(^5\) While the Kalman filter estimation procedure for affine models is well established in the case where all factors are assumed to be unobserved (see, for instance, de Jong, 2000, and Duan and Simonato, 1999), some issues remain to be settled once we incorporate observable macroeconomic factors into the state space vector. First, the state vector \( f \) needs to be updated in a way that guarantees that both the output gap and inflation are treated as observable factors. This issue is tackled in the next subsection by a proper definition of the measurement equation. Second, the conditional

\(^5\)The Kalman filter procedure is less time consuming than other algorithms like the Efficient Method of Moments (EMM) technique. For Gaussian models, a linear Kalman filter together with exact maximum likelihood (ML) estimation is optimal within the class of all linear estimators. Parameter estimators can be shown to be efficient and consistent (see Bollerslev and Wooldridge, 1992, for a proof). There is, however, one subtlety to be mentioned. Some of the factors are latent and based upon a linear prediction. These predicted state variables will enter in the conditional mean imputing errors in the likelihood function. This complication, however, does not invalidate the asymptotic properties mentioned above. Finally, Dusee and Stanton (2001) advocate the use of the linear Kalman filter above the EMM and SNP auxiliary model estimation approach (see Gallant and Tauchen, 1992).
mean and variance of the factors need to be computed without restricting the dynamics of the system. In other words, one should not prevent the possibility of imaginary eigenvalues with respect to the spectral decomposition of the matrix $K$. In this case, closed-form solutions for the conditional mean and variance of the factors are no longer available. This point is relevant for the computation of the transition equations used in the Kalman filter and is dealt with in a subsection below. Apart from the mentioned points, the Kalman filter is implemented in a standard way.

3.1 Measurement equation

The model is tested on a data set containing output gap, inflation and yields of different maturities. In order to estimate the parameters of this model, it is necessary to link the model to these variables via the measurement equation. We use the implied one-step ahead predictions for the output gap and inflation in order to the macroeconomic part of the model. With respect to the yield curve, we estimate the parameters so as to the observed yield curve as well as possible given the observed state vector. Let $\mathbf{y}(t)$ denote the vector of yields observed at time $t$ for maturities $\ell_i$, $i = 1; \ldots; m$, $\mathbf{y}(t) = (\mathbf{y}_1(t; \ell_1); \ldots; \mathbf{y}_m(t; \ell_m))'$, where each $\mathbf{y}_i(t; \ell_i)$ is defined as $\mathbf{y}_i(t; \ell_i)' \ln(p(t; \ell_i)) - \ell_i$. We then require the model’s parameter estimates to be optimized over the set of moments for both the macro variables and the yield curve. That is, we define the measurement equation as:

$$
\begin{bmatrix}
\mathbf{y}_1(t; \ell_1) \\
\vdots \\
\mathbf{y}_m(t; \ell_m) \\
\mathbf{y}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & A & \mathbf{B} \\
0 & 0 & \mathbf{e}_2 & \mathbf{e}_2 \\
0 & 0 & \mathbf{e}_2 & \mathbf{e}_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{y}(t) \\
\mathbf{y}(t) \\
\mathbf{y}(t) \\
\mathbf{y}(t)
\end{bmatrix}
+ \mathbf{e}_t ;
$$

where $\mathbf{e}_t$ is a $(n \times 1)$ column vector of zeros with a one on the $i$th row, $\mathbf{e}_t$ is a $(m+2) \times 1$ vector of multivariate normally distributed measurement errors, $\mathbf{a} = (a(\ell_1) = \ell_1; \ldots; a(\ell_m) = \ell_m)'$ and $\mathbf{B} = [\mathbf{b}(\ell_1) = \ell_1; \ldots; \mathbf{b}(\ell_m) = \ell_m]'$. Denoting $\mathbf{z}(t)$ as the left-hand side of (11), we can rewrite the measurement equation more concisely as:

$$
\mathbf{z}(t) = \mathbf{c}_z + \mathbf{H}\mathbf{f}(t) + \mathbf{e}(t) ;
$$

$$
\mathbf{E}_t (\mathbf{e}(t) \mathbf{e}'(t)) = \mathbf{R} ;
$$

3.2 Transition equation

It is standard in the term structure literature to transform the state space such that the $K$ matrix in (4) becomes diagonal. This procedure yields well-known closed form solutions for the conditional means and variances of the transformed factors (see, for instance, de Jong, 2000). Implicit in this approach is the assumption that the eigenvalues of $K$ are all real. While this may be a reasonable assumption in the case of latent factors, it is no longer once
macroeconomic aggregates are included in the state space.\textsuperscript{6} As such, transforming the system in terms of eigenvalues and eigenvectors, as is usually done in the latent factor literature, is no longer innocuous and can, therefore, not be done without possible major implications for the dynamics of the system. We refrain from doing so and opt for computing the conditional means and covariance matrix of the factors using the method presented by Fackler (2000).\textsuperscript{7} In this approach, we restrict the space of state vectors to the one that contains the non-transformed macro aggregates. The transition equation for the time interval $h = \xi t$ can then be written as:

\[ f(t + \xi t) = c(\xi t) + \xi t (f(t) - c(\xi t)) + v_{t+\Delta t}, \]

\[ E v_{t+\Delta t} = Q(\xi t); \]

where $c(\xi t)$; $\xi t$ and $Q(\xi t)$ determine the discrete time dynamics implied by the continuous time model. Using results in Fackler (2000), we have that:

\[ dc(t) = K \psi dt + K c(t) dt, \]

\[ d\xi t = K \xi t dt, \]

\[ dvec(Q(t)) = D \text{diag}(S) dt + (K - I_n + I_n - K) \text{vec}(Q(t)) dt; \]

with $D$ a $n^2 \times n$ matrix with elements $D_{i,j} = 1$ if $i = (j + 1)n + j$ and 0 otherwise. This system of ODEs can be solved to yield the required matrices for any discrete time interval $\xi t$: The ODEs are initiated by the boundary conditions $c(0) = 0$; $\xi t = I_n$, and $Q(0) = 0$.

### 3.3 The Kalman Filter algorithm

The above measurement and transition equations imply the efficiency of the standard Kalman filter estimator. Fixing the discrete time interval to the frequency of the sampled data $\xi t$, and using the the shorthand notation $c = c(\xi t)$; $\xi t = c(\xi t)$ and $Q = Q(\xi t)$; the transition equations:

\[ \hat{f}_{t+\Delta t | t}^3 = c + \xi t i \hat{f}_{t | t}^3 c, \]

\[ \hat{P}_{t+\Delta t | t} = \xi t \hat{P}_{t | t} \xi t + Q. \]

Moreover, as a by-product, the filter generates optimal inferences on the unobservable macro-

\textsuperscript{6}According to Dai and Singleton (working paper version, NBER 6128, of Dai and Singleton, 2000), “the assumption that the eigenvalues are real rules out some potentially interesting dynamics associated with complex eigenvalues”. Beaglehole and Tenney (1991) expand the class of interest rate processes to allow more dynamic possibilities. In particular, they present processes with decaying oscillatory behavior.

\textsuperscript{7}An equivalent method is presented in Dewachter et al. (2001).
economic factors, \( f_{t+\Delta t} \):

\[
\begin{align*}
  f_{t+\Delta t} = f_{t+\Delta t} + \hat{p}_{t+\Delta t} H' \hat{p}_{t+\Delta t} H' + R' z(t+\xi t) \end{align*}
\]

Note that, by construction of the matrix \( H \); observable macroeconomic factors are updated perfectly. The filtered series for the observable macroeconomic factors thus coincide with the observed ones. Finally, the estimation of the parameters governing both the macroeconomic dynamics as well as the prices of risk are estimated in a single step maximum likelihood procedure, where the log-likelihood function is defined as:

\[
\begin{align*}
  \ell(Z_{1:T}) &= \sum_{t=1}^{T} \frac{1}{2} \ln \left( \hat{p}_{t+\Delta t} H' \hat{p}_{t+\Delta t} H' + R' \right) \\
  &= \sum_{t=1}^{T} \frac{1}{2} z(t+\xi t) H' \hat{p}_{t+\Delta t} H' + R' \end{align*}
\]

where \( \text{nd} \) stands for the number of observations in the data set.

4 Estimation results

We first describe the data used and analyze the descriptive statistics of the sample series. The estimated model parameters and the implications for the dynamics of the observable and unobservable macroeconomic factors are discussed in Section 4.2. In Section 4.3, we present the discrete time monetary policy implied by our continuous time model. In Section 4.4, we evaluate the term structure and decompose the risk premia in terms of the macroeconomic factors. We present a variance decomposition analysis in Section 4.5 and a forecast performance evaluation in Section 4.6.

4.1 Data

We base our analysis on yield data from McCulloch and Kwon (1993) and Bliss (1997) provided by Duffee (2002). This data set consists of month-end yields on zero-coupon U.S. Treasury bonds with maturities of 3 and 6 months and 1, 2, 5, and 10 years. We use a quarterly frequency in the construction of the time series in order to incorporate the output gap series. Our data set consists of 140 data points (1964:Q1 to 1998:Q4) for each of the series. We thank Gregory Duffee for making the data available on his website. The model was also estimated for the sub-periods 1964:Q1 to 1979:Q2 (pre-Volcker) and 1982:Q1 to 1998:Q4. Although the estimates for these sub-periods differ from the ones obtained for the whole sample period, the qualitative features of the results do not depend qualitatively on the sub-period. The results are available upon request.
constructed the output gap series based on data provided by the Congressional Budget Office.\textsuperscript{10} Inflation was constructed by taking the yearly percentage change in the CPI index, that is \( \frac{1}{4} \ln CPI_t - \ln CPI_{t-4} \). The CPI index is from the International Financial Statistics (IFS) database provided by the International Monetary Fund (IMF). The series for the output gap and inflation can be seen in the top panels of Figure 1. The term structure data is shown in Figure 5 together with the results of the model.

In Table 1 we give some descriptive statistics of the sample series. The average term structure series displays an increasing yield curve and the observed variance of the term structure tends to decrease with maturity. Also, strong autocorrelation is observed in all series over the sample period. There is evidence against normality in most series in terms of skewness and excess kurtosis (both decreasing with maturity) and in terms of a summary Jarque-Bera statistic (corresponding p-values are reported in the table). Most interestingly, however, is the correlation matrix showing extreme correlation among the various bonds and more moderate correlations between bonds on the one side and output gap or inflation on the other side. All those correlations are statistically significant at a 5% level. The strong correlation between bonds (decreasing with the maturity difference) suggests the presence of a few important factors driving the yield curve. While these factors may also be part of the set of driving processes in the output gap and inflation, the lower degree of correlation suggests that the links between these macro-aggregates and the yield curve is significantly smaller.

Insert Table 1

4.2 Macroeconomic dynamics

The parameter estimates can be found in Table 2.\textsuperscript{11} Our statistical analysis confirms the attracting properties of the central tendencies for all macroeconomic factors (the diagonal elements of the matrix \( K \) are all significant different from zero). All the estimated interaction terms (off-diagonal elements in \( K \)) are also statistically significant at a 5% level. Note as well the very low mean reversion for the central tendency of inflation (see low value for \( \kappa \)). This allows us to interpret this factor as a very close approximation to the long-run expectation of inflation.\textsuperscript{12} The filtered time series for the five factors involved: two observable (output gap, \( y \), and inflation, \( \pi \)) and three non-observable (the real interest rate, \( \frac{1}{2} \pi \) and real interest rate).

\textsuperscript{10}We thank the referee for suggesting this approach. The model was also estimated making use of a proxy for the output gap obtained by using a Hodrick-Prescott (HP) filter on the GDP series over the sample period (with a standard "lambda" in the filtering procedure equal to 1600). The qualitative results are not altered.

\textsuperscript{11}The full model was estimated in a single step procedure. Optimization was performed using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm with a convergence tolerance for the gradient of the estimated coefficients equal to 1e-06. We checked the robustness of the "optimum" reported by checking convergence from an array of starting points. Note that, given the large amount of parameters to be estimated, identification and checking of the optimum is a painstaking operation. During our experiments, we found this optimum to be rather stable.

\textsuperscript{12}Dewachter and Lyrio (2003) impose the martingale assumption on the inflation central tendency factor. In this way, the mentioned factor is mathematically defined as the long-run inflation expectation. The empirical results seem, however, to reject such assumption. We opt, therefore, for a more flexible approach.
rate central tendencies, $\frac{1}{4}$ and $\frac{1}{2}$, respectively) can be seen in Figure 1.

Insert Table 2

Insert Figure 1

In order to analyze the dynamics of the system, we present in Figure 2 the impulse responses for the output gap, inflation, and the real interest rate due to a one standard deviation shock in each of the sources of uncertainty.\textsuperscript{13} All figures are presented as deviations from the baseline case. The first column of graphs gives the impulse responses of the output gap to each of the shocks. The output gap decreases, although marginally, due to a temporary excess real interest rate. It increases due to shocks in the central tendency of inflation while it decreases in response to observable inflation shocks. Inflation responses can be seen in the second column of graphs. A temporary demand shock (i.e. output in excess of the equilibrium capacity, $y > 0$) induces additional inflationary pressure while an excessively high real interest rate tends to reduce the inflationary pressure. Finally, the real interest rate responses (third column of graphs) to each of the shocks illustrate the monetary policy being implemented by the central bank. The real interest rate seems to decrease due to a shock in the output gap. The monetary authority seems, furthermore, to react in different ways depending whether it identifies a shock in inflation or in its central tendency. Real interest rates increase due to a shock in the central tendency of inflation (a possible cost-push shock). In other words, the monetary authority tends to light shocks with a long duration. Inflation shocks (possible demand shocks) being less inert cause a decrease in the real interest rate.

Insert Figure 2

The impulse responses mentioned above show in most cases a cyclical pattern, reflecting the presence of imaginary components in the eigenvalue-eigenvector decomposition of the matrix $K$. This decomposition is shown in Table 3. We observe that (the real part of) all eigenvalues are negative, implying a stationary system under the historical probability measure $P$.\textsuperscript{14} Each macro factor ($f(t)$) can be interpreted as a combination of the transformed factors ($f_T(t)$), $f(t) = V f_T(t)$, where $V$ is the matrix of right eigenvectors of $K$. The output gap depends mostly on the first two transformed factors, which have a halving time of 3 years. Inflation depends mostly on the fourth factor, which represents the central tendency of inflation and has a halving time of almost 194 years. Finally, the real interest rate is mainly a function of the third transformed factor, with a very short halving time and of the last factor, with a halving time of approximately 1.5 years.

Insert Table 3

\textsuperscript{13} The impulse responses for the central tendencies of inflation and the real interest rate are not presented since these are exogenous processes, only responding to their own shocks.

\textsuperscript{14} The system is also stationary under the risk-neutral probability measure, $Q$. The real part of all eigenvalues of the matrix $K$ are, therefore, also negative.
In order to corroborate the estimated dynamics of our model, we compare our results with the inflation forecasts from the Survey of Professional Forecasters provided by the Federal Reserve Bank of Philadelphia. Figures 3 and 4 show the average one-year and ten-year ahead inflation forecast from this survey compared with the ones implied by our model. Our estimates seem to track the patterns presented in the survey forecasts rather well. The correlation between our model forecast and the survey forecast is equal to 0.86 and 0.83 for the one-year and ten-year forecast, respectively. An OLS regression of the inflation forecasts provided by the mentioned survey ($\hat{\pi}$) on the forecasts computed based on our model ($\hat{\pi}_t$) and a constant gives the following results. Despite the fact that the model tracks the general tendencies of the survey expectations reasonably well, we reject the unbiasedness hypothesis:

\[
\begin{align*}
\text{One-year forecast: } \quad \hat{\pi}_t &= 0.013 + 0.70 \, \frac{1}{4} + \hat{\pi}_{t-1} \\
&= 0.002 (0.04) \\
R^2 &= 0.73 \\
\text{(18)} \\
\text{Ten-year forecast: } \quad \hat{\pi}_t &= 0.004 + 0.83 \, \frac{1}{4} + \hat{\pi}_{t-1} \\
&= 0.003 (0.06) \\
R^2 &= 0.69 \\
\text{(19)}
\end{align*}
\]

Insert Figures 3 and 4

4.3 Monetary policy

As in Dewachter and Lyrio (2003), we present the discrete time nominal policy rule implied by our continuous time model. The rule is expressed in terms of a weighted average of backward-looking components. These are the nominal interest rate in the previous period (smoothing component) and a central bank target including both observable and unobservable macroeconomic factors. Based on the estimates from our model, this rule can be written as:

\[
i_t = 0.75 \, i_{t-1}^{CB} + 0.25 \, i_{t-1} + \hat{A}_t
\]

where $\hat{A}_t$ is an i.i.d. shock and the central bank target is given by:

\[
i_{t-1}^{CB} = 0.0010 + 0.056 \, y_{t-1} + 0.001 \, \frac{1}{4}_{t-1} + 0.999 \, \frac{1}{4}_{t-1} + 0.929 \, \frac{1}{2}_{t-1},
\]

\[
= 0.0004 (0.018) (0.093) (0.092) (0.006)
\]

where the standard errors are presented in brackets.

Our results indicate a rather low value for the smoothing component in comparison with results reported in the literature. Using a similar model, Values around 0.8 are not unusual in the literature when making use of quarterly data. Our estimates, however, support the findings by Rudebusch (2002) who point out that the high values usually reported in the literature are probably due to missing factors in the macro dynamics. In our case, the inclusion of unobservable central tendencies might have played a significant role in decreasing the observed interest rate inertia. The significance of these central tendencies also show that the standard Taylor rule in terms of only observable variables does not hold within this framework. In fact,
the central bank target has almost a one-to-one reaction to both the inflation central tendency and the real interest rate central tendency. A positive output gap also induces an increase in the target. The reaction to observed inflation is positive but not significant.

4.4 Term structure implications

We analyze the performance of the model in fitting the term structure of interest rates and the influence of each factor in the overall fit. As can be seen from Figure 5, the model is able to fit the term structure rather well, especially for long-term maturities. The maximum standard deviation of the measurement error equals 15 basis points for the one-quarter bond yield (see diagonal elements of matrix $R$ in Table 2). Table 4 presents some additional statistics concerning the yield curve fit. The average yield curve based on the macro factors presents a good fit to the empirical average yield curve. The same applies to the volatility of the term structure.

Table 4 also reports the implied average risk premium for each bond yield (last column). It increases with the maturity, ranging from 1% to 3%. These results are in line with values reported in the latent factor literature. The risk premium also shows substantial time variation for all maturities, as can be seen in Figure 6. A decomposition of the risk premium into its different components is presented in Figure 7. All factors, except the output gap, have a significant contribution to the total risk premium. The average value for each of the components is also presented in Table 4. They are all increasing with the maturity of the bond yield. The average inflation risk premium, for example, increases with the maturity, ranging from 0.26% for the 3-month bond yield to 0.45% for the 10-year bond yield. These values are reasonably close to the estimates reported by Buraschi and Jiltsov (2001) for a similar sample period. Finally, all the estimated time-varying risk premia parameters ($\gamma$) are individually highly significant. With respect to the constant risk premia parameters, two out of the four estimated parameters are also statistically significant.

The yield curve is a feature in the state space vector and the loadings for the various maturities with respect to each factors can be seen in Figure 8. In contrast with the latent factor literature, we do not...nd evidence in favor of a standard level effect. Instead, we...nd a clear division between macro factors and its central tendencies. Macro factors ($y, \frac{1}{4}$ and $\frac{1}{2}$) influence

15 Note that the estimated dynamics are stable. As such, this policy rule, together with the macroeconomic dynamics, guarantees determinacy of the inflation process.

16 We follow Dai and Singleton (2001) and Duarte (2002) and restrict the set of estimated market prices of risk. In this way, we avoid identification problems, allowing enough flexibility in the time variation of the risk premia.

17 In order to save on space, the constant factor loadings ($a$) are not presented here. The values are available upon request.
almost exclusively the very short end of the yield curve, with the output gap having basically no influence over the entire yield curve. The sensitivities of inflation and the real interest rate decay rapidly with the time to maturity of the bond such that these factors can be clearly linked to the slope of the yield curve. The central tendencies of inflation and the real interest rate, on the other hand, affect almost the whole term structure in a significant way (except, by construction, for the very short run). Yields with maturities over one year seem to be mainly responsive to changes in these central tendencies. These central tendencies have, however, different effects on the yield curve. The inflation central tendency has a nearly identical effect on bonds with maturities above two years, constituting the factor more similar to a typical level factor. The real interest rate central tendency exhibits a strong hump-shaped effect, affecting most strongly the intermediate maturities (from 6 months to about 2 years).

Insert Figure 8

The effect of each macro shock on the term structure is illustrated by means of the impulse responses shown in Figure 9. Each graph in this figure presents the evolution of the yield curve after a shock of one standard deviation on each factor. Each curve represents the yield curve as a deviation from the baseline case after a certain time period (1, 4, 12, and 20 quarters). The yield curve responses to shocks on \( y \), \( \frac{1}{2} \), and \( \frac{3}{2} \) are marked by an initial upward tilting of the short end of the curve, being most pronounced for a real interest rate shock. Shocks to the central tendencies, however, have a much higher impact on the term structure. A shock to the central tendency of inflation generates almost a level shift of the entire yield curve. This is due to the high degree of inertia of this factor. Moreover, this type of shock has long-lasting effects. Even five years after the shock, no noticeable convergence towards the benchmark can be detected. Finally, a shock to the real interest rate central tendency has also a significant impact on the yield curve. Its effect is more pronounced for intermediate maturities but still significant for long-term bonds. In contrast to shocks to the central tendency of inflation, the overall effect on the yield curve decreases significantly with time.

Insert Figure 9

4.5 Variance decomposition

We perform a variance decomposition on the shocks related to each of the factors in the model. In this way, we identify what type of shock is most important in moving the yield curve. A general variance decomposition of yield curve changes over a horizon \( h \) can be performed by decomposing the variance-covariance matrix into the responses to each of the different shocks. Defining a yield curve shock by \( \mathbf{\xi}^h \mathbf{\gamma}(t)^{\prime} \mathbf{\gamma}(t+h) \mathbf{\gamma}(t); \) it can be shown that the variance-
covariance matrix of these shocks takes the form:

\[ E_t \mathbb{E}^h \gamma(t) \mathbb{E}_t \mathbb{E}^h \gamma(t) \mathbb{E}_t \mathbb{E}^h \gamma(t) \mathbb{E}_t = \]

\[
\begin{align*}
2 & \int_0^T \exp(K(s)) E_t (SS') \exp(K(s))' ds^5 B' = \\
& \begin{pmatrix}
3 \\
B 4 \\
\end{pmatrix} \end{align*} \]

The volatility of a given bond yield over a specific horizon can be decomposed into the contributions of each of the orthogonal shocks. More specifically, the contribution of shocks of type \( j \) in the variance of maturity \( i \) over a time interval \( h \) is given by:

\[
3 \int_0^T B_k(i) (\exp(K(s)))_{k,j} ds: \]

(22)

Table 5 presents the steady state variance decomposition for different horizons of the yield curve in terms of the five possible shocks. The main observations are as follows: (i) Overall, only three shocks seem to have a significant impact on the yield curve. These are the shocks to the real interest rate and to the central tendencies of inflation and the real interest rate. Shocks to the output gap and inflation explain, respectively, at most 0.2% and 6.2% of the variability of yield curve movements; (ii) For an infinitesimal horizon, each of the three mentioned shocks play an important role in different segments of the yield curve. Real interest rate shocks dominate the short end of the yield curve (up to 2 quarters), shocks to the central tendency of the real interest rate are predominant at medium-term maturities, while shocks to the central tendency of inflation are responsible for most of the variability at the end of the yield curve; (iii) For very long time horizons (unconditional), shocks to the central tendency of inflation are responsible for almost 100% of the variability of the yield curve; (iv) Independent of the time horizon, the variability of long-term bonds (e.g., 10 years) is mainly explained by shocks to the central tendency of inflation; (v) The importance of real interest rate shocks decreases significantly with the time horizon. For a one-year horizon, its maximum contribution is reduced to 15% for yields with maturity of one quarter; and (vi) For horizons between one and five years, shocks to both the inflation and the real interest rate central tendencies are relevant. Shocks to the latter have a higher impact on short- to medium-term bonds while shocks to the former are more important to long-term yields.

Insert Table 5

4.6 Forecast evaluation

We analyze the forecasting performance of the macro model at various horizons. We adopt the root mean square error (RMSE) as our measure of performance. For the output gap and inflation, we compare the forecasting behavior of the model against three alternatives: the random walk model, an AR(1) model, and a VAR(1) representation in output gap, inflation
and the three-month interest rate. The top panels in Figure 10 show the ratio between the RMSE of the model and each of the mentioned alternatives for a prediction horizon ranging from 1 to 40 quarters. The results indicate that the model has some predictive power. For the output gap, the alternative of no predictive power (the random walk model) is outperformed for the whole range of prediction horizons, while for inflation it outperforms the random walk except for horizons between 4 and 7 years. Also, for forecasting horizons of about 3 years, the model performs better than standard AR(1) representations for both the output gap and inflation. Against a VAR(1) representation, the model performs consistently worse in predicting both the output gap and inflation.

We also analyze the forecast performance for future yield curve evolutions. Here we take as a relevant benchmark the random walk model as it has been shown in the literature that this alternative typically outperforms the predictions based on latent multi-factor models. As can be seen in Figure 10 (lower-left panel), for short-term predictions we face the same difficulties in outperforming the random walk model as standard multi-factor models. Note, however, that alternative representations including macro variables (as in Ang and Piazzesi, 2003) also report this prediction failure at short horizons. Nevertheless, for longer prediction horizons, depending as well on the maturity being predicted, the model starts to outperform the random walk model. Finally, we take the forecast performance of a latent three factor Vasicek model as a benchmark. The results are presented in the lower-right panel of Figure 10. Except for the 10-year bond yields, our model outperforms this benchmark for horizons above one year. For some yields, this performance decreases for horizons above approximately 7 years.

5 Conclusions

We present a methodology to estimate a continuous time model of the term structure of interest rates that incorporates both observable and unobservable factors. In contrast to extant models, each of the latent factors has a macroeconomic interpretation. As such, it is well suited to tackle questions related to the interrelations between financial markets and the macroeconomy.

The proposed model is able to describe in an accurate way the joint dynamics for the macroeconomy and the yield curve. We find that the observed output gap and inflation alone do not have the dynamic properties required to fit the long end of the term structure of interest rates. This role is fulfilled by the central tendencies of inflation and the real interest rate. These central tendencies also play an important role in the description of the interest rate policy rule. The time-varying risk premia are in accordance with reported values in the literature. The inflation forecasts implied by the model seem also in line with survey data. The results also show that the variability across the yield curve is mostly explained by three types of shocks. Shocks to the real interest rate and to the central tendencies of inflation and the real interest rate have a significant influence at the short end of the term structure. The role of each of these shocks depends on the time horizon. The variability at the long end of
the yield curve is mostly explained by shocks to the central tendency of inflation. The model also proves to have forecasting power. It outperforms the random walk model for both the output gap and inflation for most of the horizon investigated and the AR(1) for horizons up to 2-3 years. It does, however, worse than a VAR(1) representation. Finally, in forecasting, the macroeconomic term structure model performs well relative to the standard latent factor model. More specifically, the model with macroeconomic factors, does, in general, a better job in predicting the term structure when compared with a three-factor latent Vasicek model.
References


Table 1: Summary statistics for the data used (1964:Q1-1998:Q4)

<table>
<thead>
<tr>
<th></th>
<th>yield_{1q}</th>
<th>yield_{2q}</th>
<th>yield_{1yr}</th>
<th>yield_{2yr}</th>
<th>yield_{5yr}</th>
<th>yield_{10yr}</th>
<th>y</th>
<th>¼</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean (%)</td>
<td>6.522</td>
<td>6.778</td>
<td>7.009</td>
<td>7.252</td>
<td>7.564</td>
<td>7.770</td>
<td>-1.261</td>
<td>4.776</td>
</tr>
<tr>
<td>Std. (%)</td>
<td>2.624</td>
<td>2.660</td>
<td>2.615</td>
<td>2.525</td>
<td>2.405</td>
<td>2.324</td>
<td>2.625</td>
<td>2.844</td>
</tr>
<tr>
<td>Auto</td>
<td>0.986</td>
<td>0.987</td>
<td>0.989</td>
<td>0.992</td>
<td>0.995</td>
<td>0.997</td>
<td>0.958</td>
<td>0.993</td>
</tr>
<tr>
<td>Skew</td>
<td>1.333</td>
<td>1.316</td>
<td>1.201</td>
<td>1.175</td>
<td>1.098</td>
<td>0.917</td>
<td>-0.469</td>
<td>1.199</td>
</tr>
<tr>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.023)</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.001)</td>
<td>(0.003)</td>
<td>(0.022)</td>
<td>(0.189)</td>
<td>(0.249)</td>
<td>(0.063)</td>
<td></td>
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<tr>
<td>JB</td>
<td>59.188</td>
<td>58.550</td>
<td>44.782</td>
<td>41.284</td>
<td>33.365</td>
<td>21.351</td>
<td>6.467</td>
<td>36.971</td>
</tr>
<tr>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.000)</td>
<td>(0.039)</td>
<td>(0.000)</td>
<td></td>
</tr>
</tbody>
</table>

Correlations

| yield_{1q} | 1.000 |
| yield_{2q} | 0.995** | 1.000 |
| yield_{1yr} | 0.982** | 0.994** | 1.000 |
| yield_{2yr} | 0.959** | 0.973** | 0.990** | 1.000 |
| yield_{5yr} | 0.900** | 0.914** | 0.943** | 0.979** | 1.000 |
| yield_{10yr} | 0.852** | 0.866** | 0.899** | 0.947** | 0.991** | 1.000 |
| y | -0.339** | -0.351** | -0.387** | -0.466** | -0.576** | -0.634** | 1.000 |
| ¼ | 0.680** | 0.683** | 0.661** | 0.617** | 0.558** | 0.533** | -0.337** | 1.000 |

The bond yield data are based on data from McCulloch and Kwon (1993) and Bliss (1997) provided by Duvee (2002) and concern U.S. Treasury bonds with maturities of 3 and 6 months and 1, 2, 5, and 10 years. Output gap (y) and inflation (π) data are constructed as mentioned in the text. The data series cover the period from 1964:Q1 until 1998:Q4, totalling 140 quarterly observations. Mean denotes the sample arithmetic average, Std standard deviation, Min minimum, Max maximum, all expressed as p.a. percentage, Auto the rst order quarterly autocorrelation, Skew and Kurt stand for skewness and kurtosis, respectively, while underneath these statistics are the significance levels at which the null of no skewness and no excess kurtosis may be rejected. JB stands for the Jarque-Bera normality test statistic with the significance level at which the null of normality may be rejected underneath it. ** indicates that the correlation is statistically significant at the 5% level.
Table 2: Maximum likelihood estimates (1964:Q1-1998:Q4)

<table>
<thead>
<tr>
<th></th>
<th>$y$</th>
<th>$\pi$</th>
<th>$\rho$</th>
<th>$\pi^*$</th>
<th>$\rho^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_y$</td>
<td>-0.3146</td>
<td>-1.0748</td>
<td>-0.4555</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0638)**</td>
<td>(0.1765)**</td>
<td>(0.1711)**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_\pi$</td>
<td>0.3854</td>
<td>-0.2452</td>
<td>-0.1319</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0640)**</td>
<td>(0.0446)**</td>
<td>(0.0249)**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_{\rho}$</td>
<td>-0.0685</td>
<td>-5.1575</td>
<td>-5.3035</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0197)**</td>
<td>(0.4009)**</td>
<td>(0.3718)**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\kappa_{\pi^*}$</td>
<td></td>
<td>-0.0036</td>
<td></td>
<td></td>
<td>-0.4849</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.0012)**</td>
<td></td>
<td></td>
<td>(0.0347)**</td>
</tr>
<tr>
<td>$\kappa_{\rho^*}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\theta$.  

| $\lambda$ | -63.9951 | 34.8309 | 32.5392 | -21.9563 |
|           | (65.1136) | (63.3615) | (14.2244)** | (11.0275)** |
| $\psi_{\rho}$ | 0.0082    | -0.3672  | -0.9849  | -1.1884  |
|           | (0.0030)** | (0.1068)** | (0.2204)** | (0.3716)** |
| $\sigma^2$ | 0.000279  | 0.000146 | 0.001545 | 0.000067 |
|           | (0.000043)** | (0.000019)** | (0.0000417)** | (0.000010)** |

ML estimates with robust standard errors underneath. Only the lower diagonal of the measurement error covariance matrix is given. These values are multiplied by $10^6$. Total loglikelihood amounts to 5881.6885 or 42.0121 on average (excluding the constant in the loglikelihood). ** and * indicate that the value is statistically significant at the 5% and 10% level, respectively.
Table 3: Diagnostic statistics of the estimated model

<table>
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<th>Factor</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>-0.228 + (0.472)</td>
<td>-0.228 - (0.472)</td>
<td>-5.406</td>
<td>-0.004</td>
<td>-0.485</td>
</tr>
<tr>
<td>Halving time (yr)</td>
<td>3.03</td>
<td>3.03</td>
<td>0.13</td>
<td>193.53</td>
<td>1.43</td>
</tr>
<tr>
<td>Eigenvector</td>
<td>y</td>
<td>0.604 + (0.182)</td>
<td>0.604 - (0.182)</td>
<td>0.189</td>
<td>-0.008</td>
</tr>
<tr>
<td></td>
<td>¼</td>
<td>0.028 - (0.489)</td>
<td>0.028 + (0.489)</td>
<td>0.038</td>
<td>1.004</td>
</tr>
<tr>
<td></td>
<td>½</td>
<td>0.009 + (0.494)</td>
<td>0.009 - (0.494)</td>
<td>2.019</td>
<td>-0.004</td>
</tr>
<tr>
<td></td>
<td>¼</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>½</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</table>

The imaginary part of the eigenvalues are presented in brackets.

Table 4: Diagnostic statistics of the estimated model

<table>
<thead>
<tr>
<th></th>
<th>Mean (%)</th>
<th>Volatility (%)</th>
<th>Average empirical risk premium (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>data (%)</td>
<td>emp. data (yr)</td>
<td>emp. cte (yr) y ¼ ½ ¼ ½ total</td>
</tr>
<tr>
<td>y</td>
<td>-1.261</td>
<td>-1.261</td>
<td>2.625 2.625</td>
</tr>
<tr>
<td>¼</td>
<td>4.776</td>
<td>4.776</td>
<td>2.844 2.844</td>
</tr>
<tr>
<td>½</td>
<td>1.747</td>
<td>1.302</td>
<td>2.198 2.224</td>
</tr>
<tr>
<td>¼</td>
<td>4.519</td>
<td></td>
<td>1.647</td>
</tr>
<tr>
<td>½</td>
<td>1.548</td>
<td></td>
<td>1.334</td>
</tr>
<tr>
<td>yield_1yr</td>
<td>6.522</td>
<td>6.517</td>
<td>2.624 2.630 -0.80 0.0016 0.26 0.19 0.81 0.36 0.83</td>
</tr>
<tr>
<td>yield_2yr</td>
<td>6.778</td>
<td>6.761</td>
<td>2.660 2.644 -0.99 0.0021 0.35 0.26 1.07 0.47 1.17</td>
</tr>
<tr>
<td>yield_1yr</td>
<td>7.009</td>
<td>7.015</td>
<td>2.615 2.618 -0.95 0.0023 0.39 0.28 1.19 0.52 1.45</td>
</tr>
<tr>
<td>yield_2yr</td>
<td>7.252</td>
<td>7.255</td>
<td>2.525 2.527 -0.71 0.0024 0.40 0.29 1.22 0.54 1.75</td>
</tr>
<tr>
<td>yield_3yr</td>
<td>7.564</td>
<td>7.569</td>
<td>2.405 2.405 -0.39 0.0025 0.43 0.32 1.34 0.59 2.31</td>
</tr>
<tr>
<td>yield_10yr</td>
<td>7.770</td>
<td>7.769</td>
<td>2.323 2.319 0.38 0.0026 0.45 0.33 1.38 0.61 3.15</td>
</tr>
</tbody>
</table>

All values are on an annual basis. emp denotes the empirical results from the model.
Table 5: Yield curve variance decomposition at different horizons in the steady state

<table>
<thead>
<tr>
<th>Time horizon: $h = dt$</th>
<th>$\frac{q}{2}$</th>
<th>$\frac{q}{4}$</th>
<th>$\frac{q}{8}$</th>
<th>$\frac{q}{16}$</th>
<th>$\frac{q}{32}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.024</td>
<td>6.156</td>
<td>78.130</td>
<td>3.215</td>
<td>12.474</td>
<td>100</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.062</td>
<td>2.971</td>
<td>49.357</td>
<td>10.541</td>
<td>37.069</td>
<td>100</td>
</tr>
<tr>
<td>$y_{1yr}$</td>
<td>0.067</td>
<td>0.457</td>
<td>18.017</td>
<td>21.129</td>
<td>60.331</td>
<td>100</td>
</tr>
<tr>
<td>$y_{2yr}$</td>
<td>0.011</td>
<td>0.001</td>
<td>5.687</td>
<td>32.571</td>
<td>61.728</td>
<td>100</td>
</tr>
<tr>
<td>$y_{5yr}$</td>
<td>0.090</td>
<td>0.097</td>
<td>1.761</td>
<td>57.445</td>
<td>40.607</td>
<td>100</td>
</tr>
<tr>
<td>$y_{10yr}$</td>
<td>0.079</td>
<td>0.008</td>
<td>0.665</td>
<td>80.534</td>
<td>18.713</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time horizon: $h = 1$ year</th>
<th>$\frac{q}{2}$</th>
<th>$\frac{q}{4}$</th>
<th>$\frac{q}{8}$</th>
<th>$\frac{q}{16}$</th>
<th>$\frac{q}{32}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.115</td>
<td>1.100</td>
<td>15.488</td>
<td>24.197</td>
<td>59.100</td>
<td>100</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.080</td>
<td>0.413</td>
<td>6.417</td>
<td>28.313</td>
<td>64.778</td>
<td>100</td>
</tr>
<tr>
<td>$y_{1yr}$</td>
<td>0.035</td>
<td>0.296</td>
<td>1.978</td>
<td>33.596</td>
<td>64.095</td>
<td>100</td>
</tr>
<tr>
<td>$y_{2yr}$</td>
<td>0.012</td>
<td>0.415</td>
<td>0.641</td>
<td>43.106</td>
<td>55.826</td>
<td>100</td>
</tr>
<tr>
<td>$y_{5yr}$</td>
<td>0.167</td>
<td>0.280</td>
<td>0.226</td>
<td>66.396</td>
<td>32.932</td>
<td>100</td>
</tr>
<tr>
<td>$y_{10yr}$</td>
<td>0.090</td>
<td>0.040</td>
<td>0.082</td>
<td>85.733</td>
<td>14.056</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time horizon: $h = 5$ years</th>
<th>$\frac{q}{2}$</th>
<th>$\frac{q}{4}$</th>
<th>$\frac{q}{8}$</th>
<th>$\frac{q}{16}$</th>
<th>$\frac{q}{32}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.101</td>
<td>0.585</td>
<td>6.038</td>
<td>52.364</td>
<td>40.911</td>
<td>100</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.125</td>
<td>0.347</td>
<td>2.441</td>
<td>55.142</td>
<td>41.944</td>
<td>100</td>
</tr>
<tr>
<td>$y_{1yr}$</td>
<td>0.165</td>
<td>0.312</td>
<td>0.741</td>
<td>59.617</td>
<td>39.165</td>
<td>100</td>
</tr>
<tr>
<td>$y_{2yr}$</td>
<td>0.214</td>
<td>0.305</td>
<td>0.233</td>
<td>68.122</td>
<td>31.126</td>
<td>100</td>
</tr>
<tr>
<td>$y_{5yr}$</td>
<td>0.216</td>
<td>0.123</td>
<td>0.072</td>
<td>84.782</td>
<td>14.807</td>
<td>100</td>
</tr>
<tr>
<td>$y_{10yr}$</td>
<td>0.057</td>
<td>0.017</td>
<td>0.021</td>
<td>94.642</td>
<td>5.264</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time horizon: $h = 10$ years</th>
<th>$\frac{q}{2}$</th>
<th>$\frac{q}{4}$</th>
<th>$\frac{q}{8}$</th>
<th>$\frac{q}{16}$</th>
<th>$\frac{q}{32}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.086</td>
<td>0.403</td>
<td>4.088</td>
<td>67.468</td>
<td>27.954</td>
<td>100</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.103</td>
<td>0.245</td>
<td>1.640</td>
<td>69.588</td>
<td>28.424</td>
<td>100</td>
</tr>
<tr>
<td>$y_{1yr}$</td>
<td>0.126</td>
<td>0.222</td>
<td>0.488</td>
<td>73.169</td>
<td>25.995</td>
<td>100</td>
</tr>
<tr>
<td>$y_{2yr}$</td>
<td>0.147</td>
<td>0.211</td>
<td>0.147</td>
<td>79.847</td>
<td>19.648</td>
<td>100</td>
</tr>
<tr>
<td>$y_{5yr}$</td>
<td>0.127</td>
<td>0.081</td>
<td>0.041</td>
<td>91.383</td>
<td>8.639</td>
<td>100</td>
</tr>
<tr>
<td>$y_{10yr}$</td>
<td>0.031</td>
<td>0.011</td>
<td>0.011</td>
<td>97.161</td>
<td>2.835</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Unconditional</th>
<th>$\frac{q}{2}$</th>
<th>$\frac{q}{4}$</th>
<th>$\frac{q}{8}$</th>
<th>$\frac{q}{16}$</th>
<th>$\frac{q}{32}$</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>0.009</td>
<td>0.041</td>
<td>0.412</td>
<td>96.725</td>
<td>2.814</td>
<td>100</td>
</tr>
<tr>
<td>$y_2$</td>
<td>0.010</td>
<td>0.024</td>
<td>0.162</td>
<td>97.002</td>
<td>2.802</td>
<td>100</td>
</tr>
<tr>
<td>$y_{1yr}$</td>
<td>0.012</td>
<td>0.021</td>
<td>0.046</td>
<td>97.464</td>
<td>2.456</td>
<td>100</td>
</tr>
<tr>
<td>$y_{2yr}$</td>
<td>0.013</td>
<td>0.018</td>
<td>0.013</td>
<td>98.249</td>
<td>1.706</td>
<td>100</td>
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<tr>
<td>$y_{5yr}$</td>
<td>0.010</td>
<td>0.006</td>
<td>0.003</td>
<td>99.350</td>
<td>0.631</td>
<td>100</td>
</tr>
<tr>
<td>$y_{10yr}$</td>
<td>0.002</td>
<td>0.001</td>
<td>0.001</td>
<td>99.799</td>
<td>0.197</td>
<td>100</td>
</tr>
</tbody>
</table>
Figure 1: Macro variables and their estimated central tendencies.
Figure 2: Impulse response functions.
INFLATION (1-yr Forecast)

Figure 3: Comparison of average 1-year ahead inflation forecast - Model vs. Survey of Professional Forecasters.
Figure 4: Comparison of average 10-year ahead inflation forecast - Model vs. Survey of Professional Forecasters.
Figure 5: Model of the term structure of interest rates.
TOTAL RISK PREMIUM

Figure 6: Total risk premium.
Note the different scale on the y-axis for the output gap component.

Figure 7: Factor decomposition of risk premium.
**FACTOR LOADINGS**

![Factor Loadings Graph](image)

Figure 8: Factor loadings.
Figure 9: Time evolution of yield curve after a macroeconomic shock.
Figure 10: Forecasting performance of the model relative to the random walk (RW), AR(1), VAR(1), and three-factor latent model.