Abstract

First we explain why the computation of the Hankel determinant built on the initial sequence of Bell numbers is equivalent to the determination of the maximal-length constant sequence of these numbers reduced modulo a prime p. That was the initial motivation for that research. Next we give four very different proofs of the precise value of this determinant. Two of them, by the author, are generalized to Bell polynomials. The first one results from an analytic theorem of Sylvester; the second one is a consequence of a direct factorization of the Hankel matrix. Another proof was given by Delsarte, using Charlier polynomials. The last one is due to Flajolet, using continued fractions.

Two new demonstrations were still published in 2000. Indeed, a quite new combinatorial, «enumerative» interpretation [5] was found by Richard Ehrenborg, at the time a member of the Institute for Advanced Study in Princeton. And Jet Wimp, of the Drexel University of Philadelphia, gave, amongst many other new formulas and generalizations [18], the last analytical proof... at this time.

Next, starting from our factorization method above, we give addition formulas for several numerical sequences and polynomials built on classical combinatorial sequences (Catalan, Hermite, Euler,...). These relations give also an explicit computation of Hankel determinants.
1. The initial problem

Let $B_n$ be the $n^{th}$ Bell number, i.e. the number of equivalences over a set of $n$ elements.
If $p$ is a prime number, then [8], [9]

- the sequence $(B_n)$ reduced modulo $p$ is purely periodic

- the length $k_p$ of the period divides $\frac{p^p - 1}{p - 1}$, and here we suppose moreover that $k_p = \frac{p^p - 1}{p - 1}$ (actually, that equality stands for every known case)

- in a period, there exists one, and only one, sequence $B_a \equiv B_{a+1} \equiv ... \equiv B_{a+p-2} \equiv 0 \pmod{p}$, with the starting index $a = a_p = \frac{p + (p - 2) k_p}{p - 1}$

- in a period there exists one, and only one, sequence $B_b \equiv B_{b+1} \equiv ... \equiv B_{b+p-1} \equiv c \pmod{p}$, where the starting index $b = b_p = a_p + \frac{k_p - 1}{p}$ and $c = c_p \equiv (-1)^{\frac{p-1}{2}} \det(B_{i+j})$ $(i, j \in \{0, 1, ..., p - 1\})$

Here is a direct proof of this value of $c_p$

Let us put $V_i^- = \begin{pmatrix} B_i \\ B_{i+1} \\ ... \\ B_{i+p-1} \end{pmatrix}$ and $D_{p,i} = \det(V_i^-;V_{i+1};...;V_{i+p-1})$

Knowing (*) that $B_{n+p} \equiv B_{n+1} + B_n \pmod{p}$, we see at once that $D_{p,i+1} \equiv D_{p,i} \pmod{p}$

In other terms, $D_{p,i}$ does not depend on $i$:

$\forall i, D_{p,i} \equiv D_p \pmod{p}$

Knowing also (**) that $B_{np} \equiv B_{n+1} \pmod{p}$, we can rewrite the constant $p$-sequence $B_{(b-1)p} \equiv B_{bp} \equiv ... \equiv B_{(b+p-2)p} \equiv c \pmod{p}$

Let us now substract the consecutive terms, using again (*). We get

$B_{(b-1)p+1} \equiv B_{bp+1} \equiv ... \equiv B_{(b+p-3)p+1} \equiv 0 \pmod{p}$

Iterating that operation, we get

$B_{(b-1)p+2} \equiv B_{bp+2} \equiv ... \equiv B_{(b+p-4)p+2} \equiv 0 \pmod{p}$,

and so on...

Looking to the so generated array, we see that the matrix $M$, of element $M_{ij} = B_{(b-1)p+i+pj}$ (where $i, j \in \{0, 1, ..., p-1\}$) verifies...
the asterisks denoting unknown elements.

Thus we have
\[(B_{(b-1)p}, B_{(b-1)p+1}, ..., B_{(b-1)p+2p-2}) \equiv (c, 0, ..., 0, c, 0, ..., 0) \pmod{p},\]
the first sequence of '0' being of length \(p-1\), the second one of length \(p-2\).

And the Hankel determinant
\[
D_p \equiv D_{p,(b-1)p} \equiv \begin{vmatrix} c & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & c \\ 0 & 0 & 0 & \cdots & c & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & c & \cdots & 0 & 0 \\ 0 & c & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{(p-1)/2} c^p \pmod{p}
\]

The little theorem of Fermat gives now at once
\[B_b \equiv c \equiv (-1)^{(p-1)/2} D_p \pmod{p}\]

But, again, \(D_p, s\) does not depend on \(i\), and
\[D_p \equiv D_{p,0} = \det(B_{i+j})\]

We see that the evaluation of the constant \(p\)-sequence of Bell numbers modulo \(p\) "naturally" leads to the Hankel determinant built on the initial sequence of these numbers.

N.B J.W. Layman [7] has shown the existence and unicity of both sequences without the assumption of maximal length period (the computation of \(a_p, b_p\) is then somewhat more difficult, but \(c_p\) remains the same).

We are now going to give four methods to compute this determinant.

**Method of Philippe Delsarte [4]**

The numbers of Stirling of the first kind \(s(k,i)\) are defined by
\[k! \binom{x}{k} = \sum_{i=0}^{k} s(k,i) x^i\]
The \( s(k,i) \) are the numbers of permutations of \( k \) objects, with \( i \) cycles. Their matrix is the inverse of the matrix of Stirling numbers of the second kind \( S(k,i) \), i.e. the numbers of equivalences with \( i \) classes over a set of \( k \) elements.

Of course, \( B_k = \sum_{i=1}^{k} S(k,i) \)

Delsarte introduces the coefficients

\[
c_{n,i} = (-1)^n \sum_{k=i}^{n} (-1)^k \binom{n}{k} s(k,i)
\]

In particular

\( c_{n,n} = 1 \) and \( c_{n,i} = 0 \) when \( i > n \)

Using the orthogonality of Charlier polynomials, he shows that

\[
\forall i, j \leq n, \sum_{k=0}^{n} \sum_{t=0}^{n} c_{i,k} c_{j,t} B_{k+t} = i! \delta_{ij}
\]

where, of course, \( \delta_{ij} \) is the Kronecker symbol \( \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \)

This can be rewritten as the product

\[
C_n D_n C_n^t = \begin{pmatrix} 0! & 0 & 0 & 0 \\ 0 & 1! & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & (n-1)! \end{pmatrix}
\]

\( C_n \) is the triangular matrix of the \( c_{i,j} \)
\( C_n^t \) is its transposed matrix \( \begin{cases} (i, j \in \{0, 1, ..., n-1\}) \end{cases} \)
\( D_n \) is defined by \( (D_n)_{ij} = B_{ij} \)

So, we have \( (\det C_n)^2 (\det D_n) = \prod_{k=0}^{n} k! \)

With the obvious value \( \det C_n = 1 \), this gives

\[
\det D_n = \prod_{k=0}^{n} k!
\]

Remarks

- The \( c_{n,i} \) are heavy to compute by their definition. It's better to use the recurrence [10]

\[
c_{n+1,i} = c_{n,i-1} - n c_{n-1,i} + (n+1) c_{n,i}
\]

with the initial values \( c_{1,1} = 1, c_{2,1} = -1, c_{2,2} = 1 \)
When \( p \) is a prime, our formula for the Hankel determinant gives \(^{11}\)

\[
(\det(B_{ij}))^2_{i,j=0,\ldots,p-1} = \begin{cases} 
+1 \pmod p & \text{if } p \equiv 3 \pmod 4 \\
-1 \pmod p & \text{if } p \equiv 1 \pmod 4
\end{cases}
\]

Indeed, by an immediate induction from the theorem of Wilson, we see that 
\[\forall k \in \{0, 1, \ldots, p-1\}, \ (-1)^k + k! (p - 1 - k)! \equiv 0 \pmod p.\]

So 
\[
(\det(B_{ij}))^2_{i,j=0,\ldots,p-1} = \prod_{k=0}^{p-1} k! \equiv \prod_{k=0}^{(p-3)/2} (p - 1 - k)! \equiv (\frac{p-1}{2})! \prod_{k=0}^{(p-3)/2} (-1)^{k+1} 
\]

\[
\equiv \left(\frac{p-1}{2}\right)! (-1)^{(p^2-1)/8} \equiv \left(\frac{p-1}{2}\right)! \left(\frac{2}{p}\right) \pmod p, \text{ where } \left(\frac{2}{p}\right) \text{ is the symbol of Legendre for the quadratic character of } 2 \text{ modulo } p.
\]

But Euler’s criterion gives also 
\[
\left(\frac{2}{p}\right) \equiv 2^{(p-1)/2} \pmod p.
\]

And we find 
\[
(\det(B_{ij}))^2_{i,j=0,\ldots,p-1} \equiv \prod_{k=1}^{(p-1)/2} (2k) \pmod p.
\]

Now, the little theorem of Fermat and again the formula 
\[
(-1)^k + k! (p - 1 - k)! \equiv 0 \pmod p
\]
end the proof.

In a letter (June 3rd, 1979) to the author, Pierre Deligne remarks moreover that

\[
\left(\frac{p-1}{2}\right)! \equiv \begin{cases} 
-1, & \text{if } h \equiv 1 \pmod 4 \\
+1, & \text{if } h \equiv 3 \pmod 4
\end{cases} \pmod p
\]

where \( h \) is the number of classes of \( \mathbb{Q}(\sqrt{-p}) \) (for \( p = 3 \), replace \( h \) by \( \frac{h}{3} \)).

Method of Philippe Flajolet \(^{6}\)

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a (formal) series with integer coefficients \( a_n \).

If \( f(z) \) has the (also formal) Jacobi-Stieltjes continued fraction expansion

\[
f(z) = \frac{1}{1 - \alpha_0 z - \frac{\beta_1 z^2}{1 - \alpha_1 z - \frac{\beta_2 z^2}{1 - \alpha_2 z - \frac{\beta_3 z^2}{\ldots}}}}
\]

with integer coefficients \( \alpha_i, \beta_i \), then
\[
\det(a_{ij})_{i,j \in \{0, 1, \ldots, n\}} = \prod_{k=1}^{n} M_k,
\]

with \(M_k = \prod_{i=1}^{k} \beta_i\)

Next, Flajolet shows (amongst other beautiful examples) that, if we take \(a_n = B_n\), then \(\alpha_k = k + 1\) and \(\beta_k = k\).

And the theorem follows at once.

**First method of the author [11]**

Let us first recall Sylvester's theorem.
Let \(f\) be a \((2n - 2)\)-times derivable function.
Let us define the \(D_n\) operator by
\[
D_n f = \det \begin{pmatrix}
\frac{d^{i+j}}{dz^i z^j} f \\
(i, j \in \{0, 1, \ldots, n-1\})
\end{pmatrix}
\]

Then
\[
(D_{n+1} f) (D_{n-1} f) = D_2 (D_n f)
\]

Let us also recall that the generating "exponential" series for the Bell numbers is
\[
\exp(\exp(z)-1) = \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}
\]

Applying Sylvester's theorem to this function, we get by induction
\[
D_n \exp(\exp(z)-1) = \left( \prod_{k=0}^{n-1} k! \right) \exp\left(\frac{n(n-1)}{2} z + n \exp(z) - n\right)
\]

But, by the definition of \(D_n\) itself, and knowing that
\[
\forall n > 0, \quad \frac{d^n}{dz^n} \exp(\exp(z)) = \sum_{k=1}^{n} S(n,k) \exp(kz + \exp(z)),
\]
we find, on the other hand, that
\[
D_n \exp(\exp(z)-1) = (\exp(\exp(z)-1))^n \det(S_{i+j}(z)) \quad (i, j \in \{0, 1, \ldots, n-1\})
\]

where \(S_0(z) = 1\) and \(\forall m > 0, S_m(z) = \sum_{k=1}^{m} S(m,k) \exp(kz)\)

Comparing these two results, and replacing \(\exp(z)\) by \(z\), we find the polynomial identity
\[
\det(B_{ij}(z)) = \left( \prod_{k=0}^{n} k! \right) z^{n(n+1)/2} \quad (i, j \in \{0, 1, \ldots, n\})
\]

where \( B_0(z) = 1 \) and \( B_n(z) = \sum_{k=1}^{n} S(n,k) z^k \)

The special case \( z = 1 \) gives again the theorem.

Second method of the author \([12]\)

Let us keep the same notations.

The classical relation \( S(m+1,k) = k S(m,k) + S(m,k-1) \) leads to

\[
B_{m+1}(z) = z \left( B_m(z) + \frac{d}{dz} B_m(z) \right)
\]

and next, by induction, to

\[
B_{m+n}(z) = \sum_{k=0}^{\min(m,n)} \frac{z^k}{k!} \left( \frac{d^k}{dz^k} B_m(z) \right) \left( \frac{d^k}{dz^k} B_n(z) \right)
\]

We can rewrite this identity as a product of matrices:

\[
\begin{pmatrix}
B_0(z) & B_1(z) & B_2(z) & \ldots & B_n(z) \\
B_1(z) & B_2(z) & B_3(z) & \ldots & B_{n+1}(z) \\
B_2(z) & B_3(z) & B_4(z) & \ldots & B_{n+2}(z) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_n(z) & B_{n+1}(z) & B_{n+2}(z) & \ldots & B_{2n}(z)
\end{pmatrix}
\]

\[
\left( \begin{array}{c}
\frac{z^0}{0!} \frac{d^0}{dz^0} B_0(z) \\
\frac{z^1}{1!} \frac{d^1}{dz^1} B_1(z) \\
\frac{z^2}{2!} \frac{d^2}{dz^2} B_2(z) \\
\vdots \\
\frac{z^n}{n!} \frac{d^n}{dz^n} B_n(z)
\end{array} \right) = \left( \begin{array}{c}
\frac{d^0}{dz^0} B_0(z) \\
\frac{d^1}{dz^1} B_1(z) \\
\frac{d^2}{dz^2} B_2(z) \\
\vdots \\
\frac{d^n}{dz^n} B_n(z)
\end{array} \right)
\]

But, of course, \( \frac{d^t}{dz^t} B_t(z) = t! \)

Therefore, we find again the same polynomial generalization of our theorem

\[
\det(B_{ij}(z)) = \left( \prod_{k=0}^{n} k! \right) z^{n(n+1)/2} \quad (i, j \in \{0, 1, \ldots, n\})
\]
Remark

We can avoid the induction in the proof of the addition formula

\[ B_{m+n}(z) = \sum_{k=0}^{\min(m,n)} \frac{z^k}{k!} \left( \frac{d^k}{dz^k} B_m(z) \right) \left( \frac{d^k}{dz^k} B_n(z) \right) \]

Indeed

\[ \sum_{k=0}^{\min(m,n)} \frac{z^k}{k!} \left( \frac{d^k}{dz^k} B_m(z) \right) \left( \frac{d^k}{dz^k} B_n(z) \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \left( \frac{d^k}{dz^k} B_m(z) \right) \left( \frac{d^k}{dz^k} B_n(z) \right) \]

\[ = \sum_{k=0}^{\infty} \frac{z^k}{k!} \left( \frac{d^k}{dz^k} B_m(z) \right) \left( \frac{d^k}{dz^k} B_n(z) \right) \left( \frac{d^k}{dz^k} e^{\exp(t) - 1} \right) \bigg|_{t=0, u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} \sum_{k=0}^{\infty} \frac{z^k}{k!} (e^t - 1)^k (e^u - 1)^k e^{\exp(t) + \exp(u) - 2} \right) \bigg|_{t=u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} e^{\exp(t) + \exp(u) - 2} \sum_{k=0}^{\infty} \frac{z^k}{k!} (e^t - 1)^k (e^u - 1)^k \right) \bigg|_{t=u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} e^{\exp(t) + \exp(u) - 2} e^{\exp(t) - 1} (e^u - 1) \right) \bigg|_{t=u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} e^{\exp(t+u) - 2} \right) \bigg|_{t=u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} e^{\exp(t+u) - 1} \right) \bigg|_{t=u=0} \]

\[ = \left( \frac{d^m}{dt^m} \frac{d^n}{du^n} \sum_{k=0}^{\infty} \frac{B_k(z)}{k!} (t+u)^k \right) \bigg|_{t=u=0} \]

\[ = B_{m+n}(z) \]

A theorem of Martin Aigner

In [1], Martin Aigner proves that Bell numbers give the unique sequence such that

\[ \forall n, \det(B_{i+j})_{i,j=0,...,n} = \det(B_{i+j+1}) = \prod_{k=0}^{n} k! \]
Let us see now other famous sequences.

2. The Hankel matrix, of any order, built on Catalan numbers has determinant 1

It is well known that the Catalan number \( c_n = \binom{2n}{n} \) counts the number of sequences of \( n \) integers 1 and \( n \) integers -1, with partial sums always positive.

Let us now introduce \( c_{n,k} = \frac{(2k+1)\binom{2n}{n+k}}{n+k+1} \).

Here is the array of the first \( c_{n,k} \)

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 9 & 5 & 1 & 1 & 1 & 1 & 1 \\
14 & 28 & 20 & 7 & 1 & 1 & 1 & 1 \\
42 & 90 & 75 & 35 & 9 & 1 & 1 & 1 \\
132 & 297 & 275 & 154 & 54 & 11 & 1 & 1 \\
429 & 1001 & 1001 & 637 & 273 & 77 & 13 & 1 \\
1430 & 3432 & 3640 & 2548 & 1260 & 440 & 104 & 15 & 1 \\
4862 & 11934 & 13260 & 9996 & 5508 & 2244 & 663 & 135 & 17 & 1 \\
\end{array}
\]

Because of the trivial relation \( c_{n,k} = \binom{2n}{n+k} \cdot \binom{2n}{n+k+1} \), we see that \( c_{n,k} \) counts the number of sequences of \( (n+k) \) integers 1 and \( (n-k) \) integers -1, with partial sums always positive.

Now the coefficient of \( x^{i+j+1} \) in \((1 - x)^2 \ (1 + x)^{2i+2j}\) is \( -2 \ c_{i+j} \).

On the other hand, the coefficient of \( x^a \) in \((1 - x) \ (1 + x)^{2i}\) is \( \binom{2i}{a} \frac{2i-2a+1}{2i-a+1} \).

Identifying the term \( x^{i+j+1} \) in the trivial identity
\[
(1- x)^2 \ (1 + x)^{2i+2j} = [(1 - x) \ (1 + x)^{2i}] \ [(1 - x) \ (1 + x)^{2j}] \]
and using antisymmetry of the coefficients of these polynomials, we get easily \[13\]
\[
\sum_{k=0}^{\min(i,j)} c_{i+k} c_{j-k}
\]

Another way of writing this formula is the following.

If \( A_n \) is the triangular matrix \((c_{i,k})_{i,k=0,...,n} \), if \( B_n \) is \( A_n^t \),

and if \( H_n \) is the Hankel matrix \((c_{i+j})_{i,j=0,...,n} \)

then \( A_n B_n = H_n \).

This relation implies \( \det(H_n) = \det(A_n) \det(B_n) = 1 \cdot 1 = 1 \).
3. The inverse of that matrix

Let $\alpha_{n,i,j} = (H_n^{-1})_{i,j}$. 

On one hand, we have $(1 - x) (1 + x)^{2n} = \sum_{k=0}^{n} c_{n,k} (x^{n-k} - x^{n+k+1})$. 

Dividing by $x^{n+1/2}$ and replacing next $x$ by $e^{2i\alpha}$, we obtain 

$$\sum_{k=0}^{n} c_{n,k} \sin (2k+1)\alpha = 2^n \sin \alpha \cos^n \alpha$$

On the other hand, by induction, 

$$\sin (2n+1)\alpha = \sum_{k=0}^{n} (-1)^{n+k} \binom{n+k}{n-k} 2^{2k} \sin \alpha \cos^{2k} \alpha$$

So, with the above notations, $(A_n^{-1})_{i,j} = (-1)^{i+j} \binom{i+j}{i-j}$ and therefore 

$$\alpha_{n,i,j} = (-1)^{i+j} \sum_{k=\text{max}(i,j)}^{n} \binom{k+i}{k-i} \binom{k+j}{k-j}$$

We have also 

$$\sum_{k=0}^{j} (-1)^{k+i} \binom{k+i}{k-i} c_{j,k} = \delta_{i,j}$$

4. Some consequences

$$\sum_{k=0}^{n} c_{n,k} = \binom{2n}{n}$$

$$\sum_{k=0}^{n} (-1)^{k} c_{n,k} = 0$$

$$\sum_{k=0}^{n} c_{n,k}^2 = c_{2n}$$

$$\sum_{k=0}^{n} (2k+1) c_{n,k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^{k} (2k+1) c_{n,k} = -2 c_{n-1}$$

$$\sum_{k=0}^{n} c_{n,k} c_{n,n-k} = \frac{2 (2n^2+2n+1) (4n)!}{n! (3n+2)!}$$

(Hint: use the Leibnitz rule on the Taylor coefficient $c_{n,k} = \frac{1}{(n-k)!} \left( \frac{d^{n-k}}{dx^{n-k}} ((1-x) (1+x)^{2n}) \right)_{x=0}$ )

$$\forall k, \sum_{n=k}^{\infty} \frac{c_{n,k}}{2^{2n}} = 2$$

(Hint: first $c_{n,k}$ is also a Fourier coefficient: $c_{n,k} = \frac{2^{2n+1}}{\pi} \int_{0}^{\pi} \cos^{2n} \alpha \sin (2k+1)\alpha \sin \alpha \, d\alpha$, and next 

$$\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{c_{n,k}}{2^{2n}} = \int_{0}^{\pi} \left( \sum_{n=0}^{\infty} \cos^{2n} \alpha \sin (2k+1)\alpha \sin \alpha \, d\alpha \right) = \int_{0}^{\pi} \frac{\sin (2k+1)\alpha}{\sin \alpha} \, d\alpha = \pi$$)
5. Motzkin numbers

Motzkin numbers can be defined by the recurrence law $M_0 = 1, M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k}$

while, for the Catalan numbers, we have $c_0 = 1, c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$.

The beginning of $(M_n)$ is 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, ...

There are of course many close relations between these two sequences. For instance

$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} c_k$ and $c_{n+1} = \sum_{k=0}^{n} \binom{n}{k} M_k$.

$M_n$ is the number of sums of $n$ terms chosen in $\{-1, 0, 1\}$, with the same amount of -1 and 1, and partial sums always positive.

The generating series are $\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n=0}^{\infty} c_n x^n$ and $\frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = \sum_{n=0}^{\infty} M_n x^n$.

One year after the publication of [13], Martin Aigner (see [2] and [3]) rediscovered the method given above for Catalan numbers.

He proved also that $\det(M_{i+j})_{i,j=0,...,n} = 1$ and, moreover, that

$\det(c_{i+j+1})_{i,j=0,...,n} = 1$ and $\det(c_{i+j+2})_{i,j=0,...,n} = n + 2$.

Actually let us look to the array of the $M(n,k)$ :

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
2 & 2 & 1 \\
4 & 5 & 3 & 1 \\
9 & 12 & 9 & 4 & 1 \\
21 & 30 & 25 & 14 & 5 & 1 \\
51 & 76 & 69 & 44 & 20 & 6 & 1 \\
\end{array}
\]

where $M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}$

We have $M_{m+n} = \sum_{k=0}^{\min(m,n)} M_{m,k} M_{n,k}$

Another result, to appear in the volume II of the book *Enumerative combinatorics* of Richard Stanley is also announced :

the numbers of Catalan give the unique sequence such that $\forall n, \det(c_{i+j})_{i,j=0,...,n} = \det(c_{i+j+1})_{i,j=0,...,n} = 1$.

For $\det(M_{i+j+1})_{i,j=0,1,...,n}$, Aigner found the cyclic values 1, 0, -1, -1, 0, 1 according to the remainder of $n$ modulo 6.
6. Catalan polynomials

We define now the $m^{th}$ Catalan polynomial by $P_m(x) = \sum_{k=0}^{m} c_{m,k} x^k$

We define also $H_n(x) = P_0(x) P_1(x) P_2(x) \ldots P_{n-1}(x)$

Note that $(1+x)$ divides every element of this determinant, except $P_0(x)$. Its value is therefore as simple as possible:

$H_n(x) = (1+x)^n$

The proof is quite easy.

Let us put $a_{n,r}(x) = \sum_{k=r}^{n} c_{n,k} x^{k-r}$ (in particular, $a_{n,0}(x) = P_n(x)$ and $a_{n,n}(x) = 1$)

This polynomial is the coefficient of $u^{n-r}$ in $(1-u) (1+u)^{2n}$, i.e. in $\sum_{k=0}^{n} c_{n,k} (u^{n-k} - u^{n+k+1}) \sum_{t=0}^{\infty} u^t x^t$.

So, we have the addition formula

$P_{m+n}(x) = a_{m,0}(x) a_{n,0}(x) + (x+1) \sum_{k=1}^{\min(m,n)} a_{m,k}(x) a_{n,k}(x)$

In other words, with the corresponding above notations, the matrix $H$, i.e.

\[
\begin{pmatrix}
P_0(x) & P_1(x) & P_2(x) & P_3(x) & \ldots & P_{n}(x) \\
P_1(x) & P_2(x) & P_3(x) & P_4(x) & \ldots & P_{n+1}(x) \\
P_2(x) & P_3(x) & P_4(x) & P_5(x) & \ldots & P_{n+2}(x) \\
P_3(x) & P_4(x) & P_5(x) & P_6(x) & \ldots & P_{n+3}(x) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
P_n(x) & P_{n+1}(x) & P_{n+2}(x) & P_{n+3}(x) & \ldots & P_{2n}(x)
\end{pmatrix}
\]

is the product of $A$, i.e.

\[
\begin{pmatrix}
a_{0,0}(x) & 0 & 0 & 0 & \ldots & 0 \\
a_{1,0}(x) & (x+1) & a_{1,1}(x) & 0 & \ldots & 0 \\
a_{2,0}(x) & (x+1) & a_{2,1}(x) & (x+1) & a_{2,2}(x) & 0 & \ldots & 0 \\
a_{3,0}(x) & (x+1) & a_{3,1}(x) & (x+1) & a_{3,2}(x) & (x+1) & a_{3,3}(x) & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n,0}(x) & (x+1) & a_{n,1}(x) & (x+1) & a_{n,2}(x) & (x+1) & a_{n,3}(x) & \ldots & (x+1) a_{n,n}(x)
\end{pmatrix}
\]
The special case \( x = 0 \) is the numerical theorem above, since \( a_{m,0}(0) = P_m(0) = c_m \).

The inversion formula becomes of course

\[
\sum_{k=0}^{j} (-1)^{k+i} \left( \binom{k+i-1}{k-i-1} x + \binom{k+i}{k-i} \right) a_{j,k}(x) = \delta_{ij}.
\]

**Remark**

Since \((x^2 - 1)(x^2 + x + 1)^n = \sum_{k=0}^{n} M_{n,k} (x^{n+k+2} - x^{n-k})\), this method works also for Motzkin numbers.

Indeed, let us define Motzkin polynomials by \( M_m(x) = \sum_{k=0}^{m} M_{m,k} x^k \).

As above, let us define also \( b_{n,r}(x) = \sum_{k=r}^{n} M_{n,k} x^{k-r} \) (so \( b_{n,0}(x) = M_n(x) \) and \( M_{n,n}(x) = 1 \)). We get the

**generalization of Aigner’s result** (at the end of § 5 of this text):

\[
M_{m+n}(x) = \sum_{k=0}^{\min(m,n)} a_{m,k}(x) \ b_{n,k}(x).
\]

An immediate, but nevertheless striking consequence is \( \forall n, \forall x, \det(M_{i+j}(x))_{i,j=0,...,n} = 1 \).

We have also \( \sum_{k=0}^{n} (k + 1) M_{n,k} = 3^n, \sum_{k=0}^{n} M_{n,k} M_{n,n-k} = M_{2n,n} - M_{2n,n+2} \), and so on...

**7. A related problem**

Consider now the polynomial \( Q_n(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^k \)

We have

\[
\begin{bmatrix}
Q_0(x) & Q_1(x) & Q_2(x) & \ldots & Q_n(x) \\
Q_1(x) & Q_2(x) & Q_3(x) & \ldots & Q_{n+1}(x) \\
Q_2(x) & Q_3(x) & Q_4(x) & \ldots & Q_{n+2}(x) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Q_n(x) & Q_{n+1}(x) & Q_{n+2}(x) & \ldots & Q_{2n}(x)
\end{bmatrix} = 2^n x^{n(n+1)/2}
\]

I have already given \[14\] an analytic proof of that identity, but the following one is almost obvious.

Indeed, let us define \( r_{m,n}(x) = \sum_{a=0}^{m-n} \left( \begin{array}{c} m \\ a \end{array} \right) \left( \begin{array}{c} m \\ a+n \end{array} \right) x^a \)
We have
\[ r_{m,0}(x) = Q_m(x) \]
and
\[ \forall n > 0, r_{m,n}(x) = r_{m-1,n-1}(x) + (x+1) r_{m-1,n}(x) + x r_{m-1,n+1}(x) \] (with \( r_{m,n}(x) = 0 \) if \( n > m \))

Here are the first \( r_{m,n}(x) \):

\[
\begin{array}{ccc}
1 & 1 \\
1 & 2x + 2 \\
x^2 + x + 1 & 3x^2 + 9x + 3 \\
x^3 + 9x^2 + 9x + 1 & 4x^3 + 24x^2 + 24x + 4 \\
x^4 + 16x^3 + 36x^2 + 16x + 1 & 6x^2 + 16x + 6 \\
\end{array}
\]

By induction, we get an addition formula analogous to the previous one for Catalan polynomials:

\[
Q_{m+n}(x) = r_{m,0}(x) r_{n,0}(x) + 2 \sum_{k=1}^{\min(m,n)} x^k r_{m,k}(x) r_{n,k}(x)
\]

Writing it as a product of matrices proves the theorem at once.

Remark: the case \( x = 1 \) gives again the theorem concerning the Hankel determinant built on the central binomial coefficients, since \( Q_m(1) = \binom{2m}{m} \).

In [18], Jet Wimp gives a strong generalization in terms of polynomials of Jacobi and Gegenbauer.

8. Hermite polynomials

The generating series for these polynomials is
\[
e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n
\]

Let us introduce the polynomials \( s_{n,k}(x) = \frac{1}{k!} \left( \frac{d^n}{dt^n} t^k e^{2tx-t^2} \right)_{t=0} \)

We see at once that
\[ \forall k, s_{n+1,k}(x) = s_{n,k-1}(x) + 2x s_{n,k}(x) - 2(k+1) s_{n,k+1}(x) \]

Particular cases are \( s_{n,0}(x) = H_n(x) \), \( s_{n,n}(x) = 1 \) and \( s_{n,k}(x) = 0 \) if \( k > n \).

Here is the array of the first \( s_{m,k}(x) \):
Next, by an immediate induction, we get

\[ \frac{d^n}{dt^n} e^{2tx-t^2} = e^{2tx-t^2} \sum_{k=0}^{n} (-2)^k s_{n,k}(t) t^k \]

and the addition formula

\[ H_{m+n}(x) = \sum_{k=0}^{\min(m,n)} (-2)^k k! s_{m,k}(x) s_{n,k}(x) \]

So, the usual technique gives (see also [15]), for the corresponding Hankel determinant, the constant value:

\[ \det (H_{i+j}(x))_{i,j = 0,...,n} = \left( \prod_{k=0}^{n} k! \right) (-2)^{n(n+1)/2} \]

9. Euler numbers

The following theorem was stated for the first time in [11], as another consequence of the identity of Sylvester.

In [18], Jet Wimp gives a quite different proof based on properties of the polynomials of Pollaczek. The new one written here is to appear in [17].

Its advantage is to be very elementary. Its inconvenient is (of course...) to lose the link established by Wimp.

We can define the nth Euler number by the Taylor series for \( \frac{1}{\cos(x)} \) at the point \( x = 0 \):

\[ \frac{1}{\cos(x)} = \sum_{n=0}^{\infty} \frac{E_n}{(2n)!} x^{2n} \quad (|x| < \frac{\pi}{2}) \]

So, \( E_0 = 1, E_1 = 1, E_2 = 5, E_3 = 61, E_4 = 1385, \ldots \)

Let us define \( E_{n,k} = \left( \frac{d^{2n}}{dx^{2n}} \left( \tan^{2k}(x) \right) \right)_{x=0} \)

Here is the array of the first \( E_{n,k} \) :
Note that $E_{n,0}$ is the $n^{th}$ Euler number $E_n$. Now
\[
\frac{d^2}{dx^2} \left( \frac{\tan^{2k}(x)}{\cos(x)} \right) =
2k(2k-1) \frac{\tan^{2k-2}(x)}{\cos(x)} + (8k^2 + 4k + 1) \frac{\tan^{2k}(x)}{\cos(x)} + 2(k + 1)(2k + 1) \frac{\tan^{2k+2}(x)}{\cos(x)}
\]
So, we have also
\[
E_{n+1,k} = 2k(2k-1)E_{n,k-1} + (8k^2 + 4k + 1)E_{n,k} + 2(k + 1)(2k + 1)E_{n,k+1}
\]
and, by induction
\[
\frac{d^{2n}}{dx^{2n}} \left( \frac{1}{\cos(x)} \right) = \sum_{k=0}^{n} E_{n,k} \frac{\tan^{2k}(x)}{\cos(x)}
\]
This implies at once the "addition" formula
\[
E_{m+n} = \sum_{k=0}^{\min(m,n)} E_{m,k} E_{n,k}
\]
In other terms, the Hankel matrix $E = (E_{i+j})$ is the product of the triangular matrix of the numbers $E_{n,k}$ by its own transposed matrix.

And we get the theorem
\[
\det(E_{i+j})_{i,j=0,\ldots,n} = \prod_{k=0}^{n} ((2k)!)^2 \quad (i,j = 0,1,\ldots,n)
\]

10. Derangement polynomials

Our next example is the derangement polynomial $d_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} x^{n-k}$

Note that, indeed, $d_n(1)$ is the number of derangements, i.e. of permutations without any fixed point, over a set of $n$ elements.

Here, we introduce the polynomials $t_{n,k}(x) = \frac{1}{k!} \frac{d^k}{dx^k} d_n(x)$

Of course, $t_{n,0}(x) = d_n(x)$ and $t_{n,n}(x) = 1$

The induction gives
\[
d_{m+n}(x) = \sum_{k=0}^{\min(m,n)} k!^2 x^{2k} t_{m,k}(x) t_{n,k}(x)
\]
and, again with the same technique (see also [16]),
\[
\det (d_{i+j}(x))_{i,j=0,\ldots,n} = \left( \prod_{k=0}^{n} k!^2 \right) x^{n(n+1)}
\]
11. Involutions

Let $I_n$ be the number of involutions over a set of $n$ elements. Of course, an involution has only fixed points and transpositions as cycles. So $I_{n+1} = I_n + n I_{n-1}$, with of course $I_0 = I_1 = 1$.

This obvious remark leads to the generating series

$$e^{x+x^2/2} = \sum_{n=0}^{\infty} \frac{I_n}{n!} x^n$$

The first $I_n$ are $1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568504, 2390480, ...$

It is also well known that $\sqrt{n} \leq \frac{I_n}{I_{n-1}} \leq \sqrt{n} + 1$.

Let us define now

$$I_{n,k} = \frac{1}{k!} \left( \frac{d^n}{dx^n} (x^k e^{x+x^2/2}) \right)_{x=0}$$

We see at once that $I_{n+1,k} = I_{n,k-1} + I_{n,k} + (k+1) I_{n,k+1}$

Particular cases are $I_{n,0} = I_n$, $I_{n,n} = 1$ and $I_{n,k} = 0$ if $k > n$.

Here is the table of the first $I_{n,k}$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2 1</td>
</tr>
<tr>
<td>4</td>
<td>6 3 1</td>
</tr>
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<td>50 40 20 5 1</td>
</tr>
<tr>
<td>76</td>
<td>156 150 80 30 6 1</td>
</tr>
</tbody>
</table>

Next, by an immediate induction, we find

$$\frac{d^n}{dx^n} e^{x+x^2/2} = e^{x+x^2/2} \sum_{k=0}^{n} I_{n,k} x^k$$

So, we have here the « addition » formula

$$I_{m+n} = \sum_{k=0}^{\min(m,n)} k! \ I_{m,k} \ I_{n,k}$$

and, always with the same technique,

$$\det(I_{m,n})_{i,j=0,...,n} = \prod_{k=0}^{n} k!$$
Bibliography


[14] Christian Radoux, *Une formule combinatoire pour les polynômes Q_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k*, Bulletin de la Société Mathématique de Belgique, vol. 45, fasc. 3, série B, 1993, 269-271.


