

Exercices de mécanique 3

Courbure et torsion

Exercice 0

Exprimer le vecteur $\vec{a} = 5\vec{1}_x - 3\vec{1}_y + 4\vec{1}_z$ en coordonnées cylindriques.

Solution

Les formules de correspondance des coordonnées cylindriques aux cartésiennes sont :

$$\begin{cases} \vec{1}_r = \cos \theta \vec{1}_x + \sin \theta \vec{1}_y \\ \vec{1}_\theta = -\sin \theta \vec{1}_x + \cos \theta \vec{1}_y \\ \vec{1}_z = \vec{1}_z \end{cases}$$

Et les formules de passage des coordonnées cartésiennes aux cylindriques sont :

$$\begin{cases} \vec{1}_x = \cos \theta \vec{1}_r - \sin \theta \vec{1}_\theta \\ \vec{1}_y = \sin \theta \vec{1}_r + \cos \theta \vec{1}_\theta \\ \vec{1}_z = \vec{1}_z \end{cases}$$

Par conséquent,

$$\begin{aligned} \vec{a} &= 5(\cos \theta \vec{1}_r - \sin \theta \vec{1}_\theta) - 3(\sin \theta \vec{1}_r + \cos \theta \vec{1}_\theta) + 4\vec{1}_z \\ &= (5 \cos \theta - 3 \sin \theta) \vec{1}_r + (-5 \sin \theta - 3 \cos \theta) \vec{1}_\theta + 4\vec{1}_z \\ \text{or } \tan \theta &= -\frac{3}{5} \rightarrow \theta = -30.9638^\circ \\ \rightarrow \vec{a} &= 5.8310 \vec{1}_r + 4\vec{1}_z \end{aligned}$$

Exercice 1

Soit la courbe définie par $x = t$; $y = t^2$; $z = \frac{2}{3}t^3$.

Calculer en fonction de t : a) sa courbure b) sa torsion.

Note : cette courbe est une cubique gauche.

Solution

Méthode 1

$$\vec{r} = t \vec{1}_x + t^2 \vec{1}_y + \frac{2}{3} t^3 \vec{1}_z$$

$$\vec{r}' = \vec{1}_x + 2t \vec{1}_y + 2t^2 \vec{1}_z \rightarrow |\vec{r}'| = \sqrt{1 + 4t^2 + 4t^4} = 2t^2 + 1$$

$$\vec{r}'' = 2 \vec{1}_y + 4t \vec{1}_z$$

$$\vec{r}''' = 4 \vec{1}_z$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = 4t^2 \vec{1}_x - 4t \vec{1}_y + 2 \vec{1}_z \rightarrow |\vec{r}' \times \vec{r}''| = \sqrt{16t^4 + 16t^2 + 4} = 2(2t^2 + 1)$$

La courbure est donc $\rho = \frac{|\vec{r}'|^3}{|\vec{r}' \times \vec{r}''|} = \frac{(2t^2 + 1)^3}{2(2t^2 + 1)} = \frac{2t^2 + 1}{2}$

$$(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = 8$$

La torsion est donc $\tau = \frac{|\vec{r}' \times \vec{r}''|^2}{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''} = \frac{4(2t^2 + 1)^2}{8} = \frac{2t^2 + 1}{2}$

Conclusion : $\rho = \tau$

Méthode 2

$$\vec{r} = t \vec{1}_x + t^2 \vec{1}_y + \frac{2}{3} t^3 \vec{1}_z$$

$$\vec{r}' = \vec{1}_x + 2t \vec{1}_y + 2t^2 \vec{1}_z \rightarrow \frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = |\vec{r}'| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$$

Le vecteur unitaire tangent est donc : $\vec{1}_t = \frac{d\vec{r}}{ds} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{1}_x + 2t \vec{1}_y + 2t^2 \vec{1}_z}{1 + 2t^2}$

$$\frac{d\vec{1}_t}{dt} = \frac{(\vec{1}_x + 2t \vec{1}_y + 2t^2 \vec{1}_z)' (1 + 2t^2)^2 - (\vec{1}_x + 2t \vec{1}_y + 2t^2 \vec{1}_z) \cdot [(1 + 2t^2)']}{(1 + 2t^2)^2}$$

$$= \frac{-4 \vec{1}_x + (2 - 4t^2) \vec{1}_y + 4t \vec{1}_z}{(1 + 2t^2)^2} \rightarrow \frac{d\vec{1}_t}{ds} = \frac{\frac{d\vec{1}_t}{dt}}{\frac{ds}{dt}} = \frac{-4 \vec{1}_x + (2 - 4t^2) \vec{1}_y + 4t \vec{1}_z}{(1 + 2t^2)^3}$$

La courbure est donc : $\frac{d\vec{1}_t}{ds} = \frac{\vec{1}_n}{\rho} \rightarrow \frac{1}{\rho} = \left| \frac{d\vec{1}_t}{ds} \right| = \frac{\sqrt{16t^2 + (2 - 4t^2)^2 + 16t^2}}{(1 + 2t^2)^3}$

$$\rightarrow \frac{1}{\rho} = \frac{\sqrt{16t^2 + 4 - 16t^2 + 16t^4 + 16t^2}}{(1 + 2t^2)^3} = \frac{2\sqrt{1 + 4t^2 + 4t^4}}{(1 + 2t^2)^3} = \frac{2(1 + 2t^2)}{(1 + 2t^2)^3} = \frac{2}{(1 + 2t^2)^2}$$

$$\vec{1}_n = \rho \frac{d\vec{1}_t}{ds} = \frac{(1 + 2t^2)^2}{2} \cdot \frac{-4 \vec{1}_x + (2 - 4t^2) \vec{1}_y + 4t \vec{1}_z}{(1 + 2t^2)^3} = \frac{-2 \vec{1}_x + (1 - 2t^2) \vec{1}_y + 2t \vec{1}_z}{1 + 2t^2}$$

$$\vec{1}_b = \vec{1}_t \times \vec{1}_n = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \frac{1}{1 + 2t^2} & \frac{2t}{1 + 2t^2} & \frac{2t^2}{1 + 2t^2} \\ \frac{-2}{1 + 2t^2} & \frac{1 - 2t^2}{1 + 2t^2} & \frac{2t}{1 + 2t^2} \end{vmatrix} = \frac{2t^2 \vec{1}_x - 2t \vec{1}_y + \vec{1}_z}{1 + 2t^2}$$

$$\frac{d\vec{1}_b}{dt} = \frac{4t \vec{1}_x + (4t^2 - 2) \vec{1}_y + 2t \vec{1}_z}{(1 + 2t^2)^2} \rightarrow \frac{d\vec{1}_b}{ds} = \frac{\frac{d\vec{1}_b}{dt}}{\frac{ds}{dt}} = \frac{4t \vec{1}_x + (4t^2 - 2) \vec{1}_y + 2t \vec{1}_z}{(1 + 2t^2)^3}$$

La torsion est donc : $\frac{1}{\tau} = \left| \frac{d\vec{1}_b}{ds} \right| = \frac{2}{(1 + 2t^2)^2}$

Pour ceux qui en doutaient encore, il est clair que la méthode 1 est nettement plus simple.

Exercice 2

Pour la même courbe, donner les équations cartésiennes de la tangente, la normale et la binormale en $t = 1$.

Solution

Soit $t = 1$

$$\begin{aligned}\vec{1}_t &= \frac{\vec{1}_x + 2t\vec{1}_y + 2t^2\vec{1}_z}{1+2t^2} & \rightarrow & \vec{1}_t = \frac{\vec{1}_x + 2\vec{1}_y + 2\vec{1}_z}{3} \\ \vec{1}_n &= \frac{-2\vec{1}_x + (1-2t^2)\vec{1}_y + 2t\vec{1}_z}{1+2t^2} & \rightarrow & \vec{1}_n = \frac{-2\vec{1}_x - \vec{1}_y + 2\vec{1}_z}{3} \\ \vec{1}_b &= \frac{2t^2\vec{1}_x - 2t\vec{1}_y + \vec{1}_z}{1+2t^2} & \rightarrow & \vec{1}_b = \frac{2\vec{1}_x - 2\vec{1}_y + \vec{1}_z}{3}\end{aligned}$$

On peut facilement vérifier que ces vecteurs sont perpendiculaires l'un à l'autre.

$$\text{Par exemple : } \vec{1}_t \cdot \vec{1}_n = \frac{-2-2+4}{9} = 0$$

Exercice 3

On donne la courbe définie par $x = \cos t$; $y = 3 \sin t$; $z = 4t$
Calculer sa courbure et sa torsion.

Note : cette courbe est une hélice.

Solution.

$$\vec{r} = (\cos t; 3 \sin t; 4t)$$

$$\vec{r}' = (-\sin t; 3 \cos t; 4) \rightarrow |\vec{r}'|^2 = 8(\cos^2 t + 2)$$

$$\vec{r}'' = (-\cos t; -3 \sin t; 0)$$

$$\vec{r}''' = (\sin t; -3 \cos t; 0)$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ -\sin t & 3 \cos t & 4 \\ -\cos t & -3 \sin t & 0 \end{vmatrix} = (12 \sin t; -4 \cos t; 3) \rightarrow |\vec{r}' \times \vec{r}''|^2 = 128 \sin^2 t + 9$$

$$(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = 12$$

$$\text{Courbure } \frac{1}{\rho} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{128 \sin^2 t + 9}{[8(\cos^2 t + 2)]^{\frac{3}{2}}}$$

$$\text{Torsion } \frac{1}{\tau} = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{|\vec{r}' \times \vec{r}''|^2} = \frac{12}{(128 \sin^2 t + 9)^2}$$

Exercice 4

On donne $\vec{a} = 5t^2 \vec{1}_x + t \vec{1}_y - t^2 \vec{1}_z$ et $\vec{b} = \sin t \vec{1}_x - \cos t \vec{1}_y$

Calculer : $\frac{d}{dt}(\vec{a} \cdot \vec{b})$; $\frac{d}{dt}(\vec{a} \times \vec{b})$; $\frac{d}{dt}(\vec{a} \cdot \vec{a})$

Solution

$$\begin{aligned} 1) \frac{d}{dt}(\vec{a} \cdot \vec{b}) &= \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b} \\ &= (5t^2 \vec{1}_x + t \vec{1}_y - t^2 \vec{1}_z) \cdot (\cos t \vec{1}_x + \sin t \vec{1}_y) + (10t \vec{1}_x + \vec{1}_y - 3t^2 \vec{1}_z) \cdot (\sin t \vec{1}_x - \cos t \vec{1}_y) \\ &= (5t^2 - 1) \cos t + 11t \sin t \end{aligned}$$

ou bien

$$\begin{aligned} \frac{d}{dt}(\vec{a} \cdot \vec{b}) &= \frac{d}{dt} \left[(5t^2 \vec{1}_x + t \vec{1}_y - t^2 \vec{1}_z) \cdot (\sin t \vec{1}_x - \cos t \vec{1}_y) \right] = \frac{d}{dt} (5t^2 \sin t - t \cos t) \\ &= 10t \sin t + 5t^2 \cos t - \cos t + t \sin t = (5t^2 - 1) \cos t + 11t \sin t \end{aligned}$$

$$\begin{aligned} 2) \frac{d}{dt}(\vec{a} \times \vec{b}) &= \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b} \\ &= (5t^2 \vec{1}_x + t \vec{1}_y - t^2 \vec{1}_z) \times (\cos t \vec{1}_x + \sin t \vec{1}_y) + (10t \vec{1}_x + \vec{1}_y - 3t^2 \vec{1}_z) \times (\sin t \vec{1}_x - \cos t \vec{1}_y) \\ &= \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ 5t^2 & t & -t^2 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix} \\ &= (t^3 \sin t - 3t^2 \cos t) \vec{1}_x - (t^3 \cos t + 3t^2 \sin t) \vec{1}_y + (5t^2 \sin t - \sin t - 11 \cos t) \vec{1}_z \end{aligned}$$

Ou bien

$$\begin{aligned} \frac{d}{dt}(\vec{a} \times \vec{b}) &= \frac{d}{dt} \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ 5t^2 & t & -t^2 \\ \sin t & -\cos t & 0 \end{vmatrix} = \frac{d}{dt} \left[-t^3 \vec{1}_x - t^3 \vec{1}_y - (5t^2 \cos t + t \sin t) \vec{1}_z \right] \\ &= (-2t^2 \cos t + t^3 \sin t) \vec{1}_x - (3t^2 \sin t + t^3 \cos t) \vec{1}_y - (10t \cos t - 5t^2 \sin t + \sin t + t \cos t) \vec{1}_z \\ &= (t^3 \sin t - 3t^2 \cos t) \vec{1}_x - (t^3 \cos t + 3t^2 \sin t) \vec{1}_y + (5t^2 \sin t - \sin t - 11 \cos t) \vec{1}_z \end{aligned}$$

$$3) \frac{d}{dt}(\vec{a} \cdot \vec{a}) = \frac{d}{dt} (25t^4 + t^2 + t^6) = 100t^3 + 2t + 6t^5$$

Ou bien

$$\frac{d}{dt}(\vec{a} \cdot \vec{a}) = 2 \frac{d\vec{a}}{dt} \cdot \vec{a} = 2 (10t \vec{1}_x + \vec{1}_y - 3t^2 \vec{1}_z) \cdot (5t^2 \vec{1}_x + t \vec{1}_y - t^2 \vec{1}_z) = 100t^3 + 2t + 6t^5$$