

Classical Symmetric Top in a Gravitational Field

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Index Terms

symmetric top, gyroscope, precession, nutation, Lagrangian formulation, mathematical physics

Abstract

The analysis of the symmetrical top is a fascinating topic in classical mechanics. It is a simple system that exhibits counter-intuitive behaviour. It's famous "gravity-defying" behaviour is well known. This brief paper first explores the Lagrangian formulation of the symmetrical top in a uniform force-field (e.g. gravity). After establishing a few constraints, a dynamic model is presented. From this model, precession rates and nutation behaviour are deduced.

I. INTRODUCTION: THE LAGRANGIAN FORMULATION

Most people are familiar with the bizarre balancing behaviour of the spinning top. Why causes the heavy spinning wheel to remain perched on its pointed stem, in apparent defiance of the pull of gravity? How can we develop a mathematical model of this behaviour? This paper will detail a methodology for quantifying the mechanics of the spinning symmetric top (gyroscope in a gravitational field).

The geometry of the symmetric gyroscopic top in a gravitational field consists of a "heavy" wheel on a narrow stem whose bottom point is fixed to the origin, but is free to rotate (as shown in Figure 1).

The z' axis passes through the center of rotation of the top. The motion of the top can be expressed in terms of the so-called Eulerian angles (ϕ, θ, ψ) . The variable θ is the angle that the z' axis makes with the stationary z axis. The variable ϕ is the angle that is formed by the projection of the z' axis in the $x - y$ plane and the x axis. Counter-clockwise rotation yields positive angles. The variable ψ is the angular position around the z' axis. Since our top is circularly symmetric, the choice of origin for ψ is arbitrary. We will only be concerned with the rate of angular rotation of the wheel about z' , given by $\dot{\psi}$. Other dynamical variables of interest are the time derivatives of ϕ and θ . These variables $\dot{\phi}$ and $\dot{\theta}$ describe the precession and nutation of the top. The length of the stem is l . We assume the mass of the stem is negligible with respect to that of the wheel. The acceleration due to gravity is g and is assumed to point "downwards" along the $-z$ axis.

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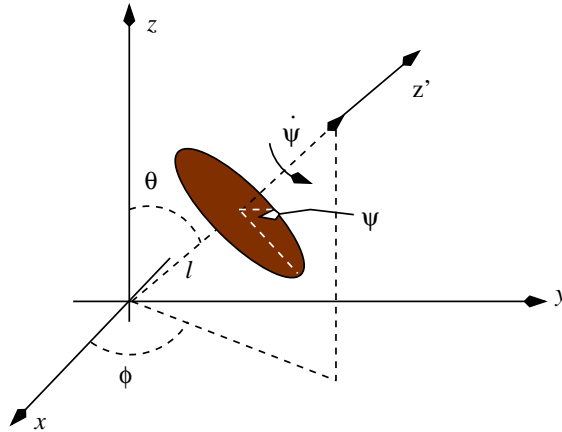


Fig. 1. Illustration of the symmetric top on a long stem. Note the orientation of the axes.

The most convenient way to derive the equations of motion for this system is to use the Lagrangian or Hamiltonian formalism. We will focus on the Lagrangian method in this paper. This formulation is readily converted to the Hamiltonian form using a simple transformation that is left to the reader to explore.

The Lagrangian of the top is deceptively simple, as seen in (1):

$$L = I_\psi (\dot{\psi} + \dot{\phi} \cos \theta)^2 + I_0 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - mgl \cos \theta \quad (1)$$

I_ψ is the moment of inertia for rotation around the z' axis ($= ma^2/2$ for a disc of radius a and mass m , ignoring the mass of the stem). I_0 is the moment of inertia of the whole structure about the x or y axis. For the wheel on a stem of length l , we have

$$I_0 = ml^2 + ma^2/4 \quad (2)$$

We see that the Lagrangian does not depend on the variables ψ , ϕ or t . As a consequence, the angular momenta p_ϕ and p_ψ and the total energy E are invariant (i.e. conserved). Utilizing the relationship

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = f_i \quad (3)$$

where x_i represents the system canonical coordinates ϕ , ψ and θ ; f_i are externally applied forces (torques), here assumed to vanish. Applying (3) to (1), we deduce the following conservation laws:

$$\frac{\partial L}{\partial \dot{\psi}} = I_\psi (\dot{\psi} + \dot{\phi} \cos \theta) = p_\psi \quad (4)$$

$$\frac{\partial L}{\partial \dot{\phi}} = I_\psi (\dot{\psi} + \dot{\phi} \cos \theta) \cos \theta + I_0 \dot{\phi} \sin^2 \theta = p_\phi \quad (5)$$

$$E = \frac{1}{2} I_\psi (\dot{\psi} + \dot{\phi} \cos \theta)^2 + \frac{1}{2} I_0 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + mgl \cos \theta \quad (6)$$

The values p_ψ , p_ϕ represent the angular momenta about the ψ and ϕ axes as well as the system total energy E . These are constants of motion and, as seen below, are indispensable quantities for deriving other aspects of the top's motion such as precession and nutation.

II. PRECESSION

The reader is probably familiar with the gyroscope's slow circling behaviour as it apparently defies gravity in its odd leaning without falling over. This is precession: movement through the azimuthal angle ϕ . For the moment, we will ignore nutation $\dot{\theta} = 0$ and focus on the leaning angle θ , assumed constant, and the rate of precession $\dot{\phi}$.

We shall use the Energy Method to derive the precession rate. This entails finding a point where the total system energy is stationary with respect to the angle θ , i.e.

$$\frac{\partial E}{\partial \theta} = 0 \quad (7)$$

We can do this because E is a conserved quantity set by the initial conditions. The energy must be a "stationary" or extreme value equal to the total system energy. Any deviation in θ from the correct value represents an error in the solution that pushes away from stationary value. This provides us with a handy mathematical trick for finding important physical properties of the system.

Since we ignore nutation (only steady precession is assumed), we eliminate the variables $\dot{\psi}$ and $\dot{\phi}$ in E by putting E in terms of the invariants using (4) and (5). We get:

$$E = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + mgl \cos \theta. \quad (8)$$

Taking the derivative of E with respect to the elevation angle θ yields

$$\frac{\partial E}{\partial \theta} = \frac{(p_\phi - p_\psi \cos \theta) p_\psi \sin^3 \theta - \sin \theta \cos \theta (p_\phi - p_\psi \cos \theta)^2}{I_0 \sin^4 \theta} - mgl \sin \theta \quad (9)$$

Solution of (9) for θ yields the angle at which the gyroscope will lean given the speed of rotation of the wheel.

The angle θ is found by solving (9) for the roots in terms of the invariant quantities. Note that (15) contains contributions from both the high speed wheel rotation about the z' axis ($\dot{\psi}$) as well as the precession motion around the z axis ($\dot{\phi}$). This is because the z and z' axes are generally not perpendicular to each other.

The precession rate is found by reorganising (5) into

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta}. \quad (10)$$

The relationship in (9) can be rearranged by grouping terms in $p_\phi - p_\psi \cos \theta$ to read

$$\frac{\cos \theta}{p_\psi \sin^2 \theta} (p_\phi - p_\psi \cos \theta)^2 - (p_\phi - p_\psi \cos \theta) + \frac{mgl I_0 \sin^2 \theta}{p_\psi} = 0. \quad (11)$$

Using the quadratic formula to solve for the roots of this equation, we have

$$p_\phi - p_\psi \cos \theta = \frac{p_\psi \sin^2 \theta}{2 \cos \theta} \left(1 \pm \sqrt{1 - \frac{4mgl I_0 \cos \theta}{p_\psi^2}} \right). \quad (12)$$

If we use (10) in (12), an explicit expression for the precession rate is produced in terms of the invariants and the angle θ :

$$\dot{\phi} = \frac{p_\psi}{2I_0 \cos \theta} \left(1 \pm \sqrt{1 - \frac{4mgl I_0 \cos \theta}{p_\psi^2}} \right). \quad (13)$$

The quadratic equation produces two solutions: the so-called “fast” precession and the “slow” precession solutions. If the rotational speed of the heavy wheel is very large (i.e. $p_\psi \gg \sqrt{4mgl \cos \theta}$), the fast precession rate is

$$\dot{\phi}_{\text{fast}} \approx \frac{p_\psi}{I_0 \cos \theta} = \frac{I_\psi \omega}{I_0 \cos \theta}. \quad (14)$$

Incidentally, the rotational speed of the wheel (usually much higher than the precession speed) is given by

$$\omega = \frac{p_\psi}{I_\psi} = \dot{\psi} - \dot{\phi} \cos \theta. \quad (15)$$

This expression is simply a recasting of (4).

The slow precession rate is found by using the small argument approximation for $\sqrt{1-x}$:

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x \quad (16)$$

Hence, we find that

$$\dot{\phi}_{\text{slow}} \approx \frac{mgl}{p_\psi} = \frac{mgl}{I_\psi \omega} \quad (17)$$

It is interesting that the slow precession rate does not depend on the “leaning angle” θ . The gyroscope will precess at a rate solely based on the wheel speed and the torque generated by the gravitational pull on the center of mass of the gyroscope. The slow precession rate is the one most commonly observed experimentally.

Now that we have an expression for the precession rate, by using (10), we can find the “lean angle” in terms of the system invariants. Solving for $\cos \theta$ in terms of $\dot{\phi}$, we get:

$$0 = (I_0 \dot{\phi} - p_\phi) + p_\psi \cos \theta - I_0 \dot{\phi} \cos^2 \theta \quad (18)$$

Using the “slow” precession rate, i.e. $\dot{\phi} = mgl/p_\psi$ in (18) and the quadratic equation to solve for $\cos \theta$, we see that

$$\cos \theta = \frac{p_\psi^2}{2I_0 mgl} \left(1 \pm \sqrt{1 + \left(\frac{4I_0 mgl}{p_\psi^2} \right) \left(\frac{I_0 mgl/p_\psi - p_\phi}{p_\psi} \right)} \right) \quad (19)$$

Only the solution with the minus sign gives us a meaningful solution, because $\cos \theta$ can only take values between -1 and 1. Moreover, if $p_\psi^2 \gg I_0 mgl$, we can use the small argument approximation for the square-root, which gives us:

$$\cos \theta \approx -\frac{I_0 mgl}{p_\psi^2} + \frac{p_\phi}{p_\psi}. \quad (20)$$

Again, since p_ψ is very large, this reduces further to

$$\cos \theta_{\text{lean}} \approx \frac{p_\phi}{p_\psi} \quad (21)$$

III. NUTATION

If we rewrite (8) including the kinetic energy of θ -directed motion (nutation), we have:

$$E = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + \frac{1}{2}I_0 \dot{\theta}^2 + mgl \cos \theta. \quad (22)$$

A “potential energy” function that provides a “restoring force” can be written as

$$V(\theta) = \frac{p_\psi^2}{2I_\psi} + \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} + mgl \cos \theta, \quad (23)$$

we can study the effects of the restoring force on the θ -directed velocity. The magnitude of this velocity is found to be:

$$|\dot{\theta}| = \sqrt{\frac{2}{I_0} (E - V(\theta))}. \quad (24)$$

The extreme values of θ can be found by solving for the roots of $E - V(\theta)$ in θ :

$$E - \frac{(p_\phi - p_\psi \cos \theta)^2}{2I_0 \sin^2 \theta} - \frac{p_\psi^2}{2I_\psi} - mgl \cos \theta = 0. \quad (25)$$

Setting $x = \cos \theta$ and reorganising into a cubic polynomial equation, we have:

$$\left\{ 2I_0 \left(E - \frac{p_\psi^2}{2I_\psi} \right) - p_\phi^2 \right\} + (2p_\phi p_\psi - 2I_0 mgl)x - \left\{ 2I_0 \left(E - \frac{p_\psi^2}{2I_\psi} \right) + p_\psi^2 \right\} x^2 + 2I_0 mgl x^3 = 0 \quad (26)$$

By solving for x in (26), we find one or two roots between -1 and 1 and another non-physical root ($|x| > 1$). The two physically meaningful roots will yield the span of angles of the nutation. If the roots are equivalent, there is no nutation; only smooth precession.

IV. EQUATIONS OF MOTION

The equations of motion are summarised below. Equation (29) can be solved using a numerical method, using (30). The remaining equations are immediately integrated based on the solution for $\theta(t)$ and the initial system state. We will not cover numerical schemes here. (perhaps this will be a topic for the future, time permitting.)

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta} \quad (27)$$

$$\dot{\psi} = \frac{p_\psi}{I_\psi} + \frac{p_\phi - p_\psi \cos \theta}{I_0 \sin^2 \theta} \cos \theta \quad (28)$$

$$I_\psi \ddot{\theta} = -\frac{\partial V}{\partial \theta} \quad (29)$$

$$\frac{\partial V}{\partial \theta} = \frac{(p_\phi - p_\psi \cos \theta) p_\psi}{I_0 \sin \theta} - \frac{\cos \theta (p_\phi - p_\psi \cos \theta)^2}{I_0 \sin^3 \theta} - mgl \sin \theta \quad (30)$$

V. EXAMPLE

Let us consider a simple example to get a feel for the order of magnitude of the motion experienced by the spinning top. Table I gives the initial state of the top as well as the various fixed parameters.

This yields the following set of parameters and invariant quantities:

$$I_\psi = 1.25 \times 10^{-4} \text{kg} \cdot \text{m}^2 \quad (31)$$

$$I_0 = 2.33 \times 10^{-3} \text{kg} \cdot \text{m}^2 \quad (32)$$

$$p_\psi = 0.0392 \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \quad (33)$$

$$p_\phi = 0.0196 \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \quad (34)$$

$$E = 12.4 \text{J} \quad (35)$$

TABLE I
TABLE OF INITIAL PARAMETERS

wheel mass	m	0.1kg
wheel radius	a	0.05m
stem length	l	0.15m
wheel spin rate	$\dot{\psi}_0$	100 π rad/s
initial precession rate	$\dot{\phi}_0$	0 rad/s
initial lean angle	θ_0	$\pi/3$ rad

If we wish to know the steady precession rate (only attainable if the initial conditions are correct such that no nutation takes place, or the nutation decays by slight frictional losses), we use (17) to find

$$\dot{\phi} = \frac{(0.1\text{kg})(9.81\text{ m/s}^2)(0.15\text{m})}{(0.0392\text{ kgm}^2/\text{s})} = 3.7\text{ s}^{-1}, \quad (36)$$

or just over a half a revolution every second.

It should be noted that this precession rate represents the average value for swept-out azimuthal angle even in the presence of nutation [1].

In order to find the range of angles θ visited by the motion of nutation, we find the roots of (26). Given our set of conditions we find one real root that gives us an angle of 1.3 radians. This indicates that the top oscillates between the original (starting) lean angle of $\pi/3 \approx 1.05$ and 1.3; that is, over a span of about 0.25 radians (or about 1/25 of a full revolution). This is a typical nutation range. The top wobbles over a fairly small angular displacement. In practice, these wobbles tend to die out as a result of slight frictional losses leaving only the rapidly spinning wheel and the precessional movement.

VI. LAST WORDS

In this brief paper we have attempted to summarise the physics of the rapidly spinning symmetric top using a classical Lagrangian approach. By identifying a set of invariants, it is possible to deduce the mechanical behaviour of the top's gyroscopic action without having to resort to full numerical (or analytic) solution of the equations of motion. Numerical solutions were not carried out for this work, but the equations of motion are easily tractable using classical time-integration methods (e.g. Runge-Kutta, Backward Euler's Method, etc.).

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